The geometry of orthogonal operators

Def Let $T \in \mathcal{L}(V)$, V a R.I.P.S., $\dim(V) < \infty$.

1) T is a rotation if T is the identity on V, or if there exists a two-dimensional subspace W of V, an orthonormal basis $\beta = \{x_1, x_2\}$ for W, and a real number θ s.t.
$T(x_1) = \cos\theta \cdot x_1 + \sin\theta \cdot x_2$
$T(x_2) = (-\sin\theta) x_1 + \cos\theta \cdot x_2$,
and $T(y) = y$ for all $y \in W^\perp$.
Then we say that T is a rotation of W about $W^\perp$, and $W^\perp$ is called the axis of rotation.

2) T is a reflection if there exists a one-dimensional subspace W of V s.t.
$T(x) = -x$ for all $x \in W$ and
$T(y) = y$ for all $y \in W^\perp$.
Then T is called a reflection of V about $W^\perp$.

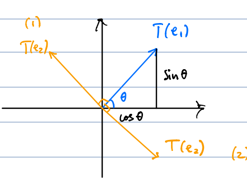
Example
1) Every rotation of $V = \mathbb{R}^2$ as discussed previously is a rotation of $W = V = \mathbb{R}^2$ about the subspace $W^\perp = \{0\}$.
2) Define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(a,b) = (-a,b)$, let $W = \mathrm{Span}(\{e_1\})$.
Then $T(x) = -x$ for all $x \in W$, and $T(y) = y$ for all $y \in W^\perp$.
Thus T is a reflection of $\mathbb{R}^2$ about $W^\perp = \mathrm{Span}(\{e_2\})$, the y-axis.

Rem 1) If $T \in \mathcal{L}(V)$ is a rotation or reflection, then T is orthogonal.
2) Moreover, if each $T_i \in \mathcal{L}(V)$ is either a rotation or a reflection, then $T = T_1 \cdot \ldots \cdot T_k \in \mathcal{L}(V)$ is orthogonal.
Proof See H/W 6.

Our next aim is to prove the converse to this: every orthogonal operator on a fin. dim. R.I.P.S. is a composition of rotations and reflections.

Example ($\dim(V) = 1$)
Let $T \in \mathcal{L}(V)$, V a R.I.P.S, $\dim(V) = 1$.
Let $x \neq 0$ be any vector in V.
Then $V = \mathrm{Span}(\{v\})$, so $T(x) = \lambda x$ for some $\lambda \in \mathbb{R}$.
Since T is orthogonal and λ is an e.val. of T, we must have $\lambda = \pm 1$. (as $\|x\| = \|T(x)\| = |\lambda|\|x\|$, so $|\lambda| = 1$).
If $\lambda = 1$, then T is the identity on V, hence T is a rotation.
If $\lambda = -1$, then $T(x) = -x$ $\forall x \in V$ by linearity of T. So T is a reflection of V about $V^\perp = \{0\}$.
So T is either a rotation or a reflection.
In the first case, $\det(T) = 1$, in the second $\det(T) = -1$.

Next we consider the case $\dim(V) = 2$.
First we understand the situation for $V = \mathbb{R}^2$.



Thm 6.23 Let $T \in \mathcal{L}(\mathbb{R}^2)$ be orthogonal.
Let β be the standard (orthonormal) basis for $\mathbb{R}^2$, and let $A = [T]_\beta$.
Then exactly one of the following is satisfied:
a) T is a rotation, and $\det(A) = 1$.
b) T is a reflection about a line through the origin, and $\det(A) = -1$.

Proof As T is orthogonal, $T(\beta) = \{T(e_1), T(e_2)\}$ is an orthonormal basis for $\mathbb{R}^2$ by Thm 6.18(c).
Since $T(e_1)$ is a unit vector, there is a unique angle θ, $0 \leq \theta \leq 2\pi$, s.t. $T(e_1) = (\cos\theta, \sin\theta)$
Since $T(e_2)$ is a unit vector and orthogonal to $T(e_1)$, there are only two possibilities for it:

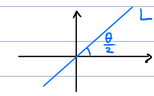
Either 1) $T(e_2) = (-\sin\theta, \cos\theta)$ or

2) $T(e_2) = (\sin\theta, -\cos\theta)$.

1) If $T(e_2) = (-\sin\theta, \cos\theta)$, then $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

From the earlier example we see that then θ is the rotation of $\mathbb{R}^2$ by the angle θ.

And $\det(A) = \cos^2\theta + \sin^2\theta = 1$.



2) Supp. $T(e_2) = (\sin\theta, -\cos\theta)$. Then $A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$.

Then T is the reflection of $\mathbb{R}^2$ about a line L through the origin, with slope $\frac{\theta}{2}$.

(Check that the def. of reflection is satisfied!). And $\det(A) = -\cos^2\theta - \sin^2\theta = -1$.

Next we need to generalize this from $\mathbb{R}^2$ to an arbitrary R.I.P.S. of dim 2.

Standard representation of I.P.S.

First recall standart presentation of vector spaces from 115A.

Def Let β be an ordered basis for an n-dimensional vector space V over the field F.
The standard representation of V w.r.t. β is the function
$\varphi_\beta : V \to F^n$ defined by $\varphi_\beta(x) = [x]_\beta$ for all $x \in V$.

Thm 2.21 For any finite-dimensional v.s. V with ordered basis β, φ_β is an isomorphism.
Moreover, for any V, W, $\dim(V) = n$, $\dim(W) = m$, and $T \in \mathcal{L}(V, W)$, if β, γ are ord. basis for V, W, respectively, then the following diagram is commutative:

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ \downarrow{\scriptstyle \varphi_\beta} & & \downarrow{\scriptstyle \varphi_\gamma} \\ F^n & \xrightarrow{L_A} & F^m \end{array}$$

That is, $L_A \varphi_\beta = \varphi_\gamma T$, i.e.
$L_A([x]_\beta) = [T(x)]_\gamma$ for all $x \in V$ (Thm 2.14)

We uprade this representation to respect inner products as well.

Thm Let V be an I.P.S. over a field F and $\dim(V) = n$.
Let β be an orthonormal basis for V. Then
$\forall x, y \in V$ we have $\langle x, y\rangle_V = \langle \varphi_\beta(x), \varphi_\beta\, y\rangle_{F^n}$,
where $\langle\,,\rangle_V$ denotes the inner product on V and $\langle\,,\rangle_{F^n}$ denotes the standard inner product on F^n.

Proof Let $\beta = \{v_1, \dots, v_n\}$.
As β is orthonormal, by Thm 6.5 we have for any $x, y \in V$:
$x = \sum_{i=1}^n \langle x, v_i\rangle v_i$, $y = \sum_{j=1}^n \langle y, v_j\rangle v_j$.
Then, by basic properties of inner products and orthogonality of β we have:
$\langle x, y\rangle_V = \sum_{i,j} \langle \langle x, v_i\rangle v_i, \langle y, v_j\rangle v_j\rangle = \sum_{i=1}^n \langle \langle x, v_i\rangle v_i, \langle y, v_i\rangle v_i\rangle = \sum_{i=1}^n \langle x, v_i\rangle \overline{\langle y, v_i\rangle}$.

But $\varphi_\beta(x) = [x]_\beta = \begin{pmatrix} \langle x, v_1\rangle \\ \vdots \\ \langle x, v_n\rangle \end{pmatrix}$, $\varphi_\beta(y) = [y]_\beta = \begin{pmatrix} \langle y, v_1\rangle \\ \vdots \\ \langle y, v_n\rangle \end{pmatrix}$, so is equal to $\langle \varphi_\beta(x), \varphi_\beta(y)\rangle_{F^n}$ by the definition of the standard inner product on F^n.

This shows that when β is an orthogonal basis, then φ_β preserves inner product as well.
Using this, we can reduce questions about general I.P.S. to the standard ones F^n.

Thm 6.45 Let V be a R.I.P.S. with $\dim(V) = 2$.
Let $T \in \mathcal{L}(V)$ be orthogonal. Then:
1) either T is a rotation, and $\det(T) = 1$, or
2) T is a reflection, and $\det(T) = -1$.

Proof.

Let β be an orthonormal basis for V.
Let $\varphi_\beta : V \to \mathbb{R}^2$ be the standard representation.
Use the previous theorem and Thm 6.23 to finish the proof (H/W 7).

Cor Let V be a 2-dimensional R.I.P.S. The composite of a reflection and a rotation on V is a reflection on V.

Pf If T_1 is a reflection on V and T_2 is a rotation on V, then by Thm 6.45:
$\det(T_1) = 1$, $\det(T_2) = -1$.
Let $T = T_2 T_1$ be the composite. As T_2 and T_1 are orthogonal operators, so is T.
And $\det(T) = \det(T_2) \cdot \det(T_1) = -1$.
By Thm 6.45, T is a reflection.
The proof for $T_1 T_2$ is analogous.

Next we study orthogonal operators on spaces of higher dimension.

Lemma Let V be a real v.s., $V \neq \{0\}$ and $\dim(V) < \infty$.
Let $T \in \mathcal{L}(V)$. Then there exists a T-inv. subspace W of V s.t. $1 \le \dim(W) \le 2$.

Proof
Fix an ordered basis $\beta = \{y_1, \dots, y_n\}$ for V.
Let $A = [T]_\beta$.
Let $\varphi_\beta : V \to \mathbb{R}^n$ be the standard representation, $\varphi_\beta(y_i) = e_i$ for $i = 1, \dots, n$.
Then φ_β is an iso. and the diagram

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "\varphi_\beta"'] & V \arrow[d, "\varphi_\beta"] \\ \mathbb{R}^n \arrow[r, "L_A"] & \mathbb{R}^n \end{tikzcd}$$

commutes, that is $L_A \varphi_\beta = \varphi_\beta T$. (see Thm 2.21 above).

In view of this, it is enough to show that there exists an L_A-invariant subspace Z of $\mathbb{R}^n$ s.t. $1 \le \dim(Z) \le 2$ (as then, taking $W = \varphi_\beta^{-1}(Z)$, W is a subspace of V with $\dim(W) = \dim(Z)$ as φ_β is an iso., and W is T-invariant as:
$\forall x \in W$, $\varphi_\beta(x) \in Z$, hence $L_A \varphi_\beta(x) \in Z$ as Z is L_A-inv, hence $\varphi_\beta T(x) \in Z$ by the diagram, that is $\varphi_\beta(T(x)) \in Z$, so $T(x) \in \varphi_\beta^{-1}(Z) = W$.)
We have $A \in M_{n\times n}(\mathbb{R}) \subseteq M_{n\times n}(\mathbb{C})$. Hence we can define $U \in \mathcal{L}(\mathbb{C}^n)$ by $U(v) = Av$ for all $v \in \mathbb{C}^n$.
By the Fundamental Thm of Algebra, every poly. over $\mathbb{C}$ has a root in $\mathbb{C}$.
In particular, the char. poly. of U has a root in $\mathbb{C}$, hence U has an e.val. $\lambda \in \mathbb{C}$ (by Thm 5.2)
Let $x \in \mathbb{C}^n$ be an e.vect. of U corresp. to λ.
We can write $\lambda = \lambda_1 + i\lambda_2$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$, and

$$x = \begin{pmatrix} a_1 + i b_1 \\ \vdots \\ a_n + i b_n \end{pmatrix} \text{ for some } a_i, b_i \in \mathbb{R}. \text{ Let } x_1 = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \ x_2 = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}. \text{ Then } x_1, x_2 \in \mathbb{R}^n \subseteq \mathbb{C}^n,$$

and $x = x_1 + i x_2$ (calculated in $\mathbb{C}^n$, where $i \in \mathbb{C}$ is a scalar).
At least one of $x_1, x_2 \neq 0$ since $x \neq 0$ as an e.vect. Hence, working in the v.s. $\mathbb{C}^n$,
$U(x) = \lambda x = (\lambda_1 + i\lambda_2)(x_1 + i x_2) = (\lambda_1 x_1 - \lambda_2 x_2) + i(\lambda_1 x_2 + \lambda_2 x_1)$.
But also $U(x) = A(x_1 + i x_2) = Ax_1 + iAx_2$.
Comparing the real and imaginary parts of these two expressions of $U(x)$ we get:
$Ax_1 = \lambda_1 x_1 - \lambda_2 x_2$ and $Ax_2 = \lambda_1 x_2 + \lambda_2 x_1$. $(*)$
Finally take $Z = \mathrm{Span}(\{x_1, x_2\})$, working in the space $\mathbb{R}^n$!
Since $x_1 \neq 0$ or $x_2 \neq 0$, Z is a non-zero subspace of $\mathbb{R}^n$, so $1 \le \dim(Z) \le 2$, and
Z is L_A-inv: given $y \in Z$, by def. of Z we have $y = a_1 x_1 + a_2 x_2$ for some $a_1, a_2 \in \mathbb{R}$, so by $(*)$
$L_A(y) = a_1 L_A(x_1) + a_2 L_A(x_2) = a_1(\lambda_1 x_1 - \lambda_2 x_2) + a_2(\lambda_1 x_2 + \lambda_2 x_1) = \underbrace{(a_1\lambda_1 + a_2\lambda_2)}_{\in \mathbb{R}} x_1 + \underbrace{(a_2\lambda_1 - a_1\lambda_2)}_{\in \mathbb{R}} x_2 \in \mathrm{Span}(\{x_1, x_2\}) = Z$.

Thm 6.46 Let V be a R.I.P.S., $V \neq \{0\}$ and $\dim(V) < \infty$. Let $T \in L(V)$ be orthogonal.
Then there exist pairwise-orthogonal, T-inv. subspaces $W_1, \dots, W_m$ of V s.t.:
a) $1 \le \dim(W_i) \le 2$ for $i = 1, \dots, m$.
b) $V = W_1 \oplus \dots \oplus W_m$.

Proof

By induction on $\dim(V)$.
If $\dim(V) = 1$ – obvious taking $W_1 = V$.
So assume Thm holds whenever $\dim(V) < n$ for some fixed integer $n > 1$.
Supp. $\dim(V) = n$.
By the lemma, there exists a T-inv. subspace W_1 of V s.t. $1 \le \dim(W_1) \le 2$.
If $W_1 = V$ – done.
Otherwise $W_1^\perp \neq \{0\}$ (as $\dim(V) = \dim(W) + \dim(W^\perp)$).
Then $W_1^\perp$ is T-inv. and $T_{W_1^\perp}$ is orthogonal (see H/W 7).
Since $\dim(W_1^\perp) = \dim(V) - \dim(W) < n$, we may apply the induction hypothesis to $T_{W_1^\perp}$, so:
there exist pairwise orthogonal T-invariant subspaces $W_2, \dots, W_m$ of $W_1^\perp$ s.t. $1 \le \dim(W_i) \le 2$ for $i = 2, \dots, n$
and $W_1^\perp = W_2 \oplus \dots \oplus W_m$.
Then $W_1, W_2, \dots, W_m$ are pairwise orthogonal and
$V \underset{\text{Thm 6.7}}{=} W_1 \oplus W_1^\perp = W_1 \oplus \dots \oplus W_m$.

We collect more information about this decomposition.

Thm 6.47' Let $T, V, W_1, \dots, W_m$ be as in Thm 6.46.
a) T_{W_i} is either a rotation or a reflection, for each $i = 1, \dots, m$.
b) The number of W_i's for which T_{W_i} is a reflection is even iff $\det(T) = 1$ and odd iff $\det(T) = -1$.
c) It is always possible to decompose V as in Thm 6.46 so that the number of W_i for which T_{W_i} is a reflection is 0 or 1, according to whether $\det(T) = 1$ or $\det(T) = -1$.
Furthermore, if T_{W_i} is a reflection, then $\dim(W_i) = 1$.

Proof

a) Each T_{W_i} is orthogonal (by H/W 7) and $1 \le \dim(W_i) \le 2$.
Then T_{W_i} is a reflection or rotation by Example 1 if $\dim(W_i) = 1$, or by Thm 6.45 if $\dim(W_i) = 2$.
b) Let r denote the number of W_i's for which T_{W_i} is a reflection.
Then, by H/W 7, $\det(T) = \det(T_{W_1}) \cdot \ldots \cdot \det(T_{W_m}) = (-1)^r$ – this gives (b).
c) Let $E = \{x \in V : T(x) = -x\}$.
Then E is a T-inv. subspace of V.
If $W = E^\perp$, then W is T-inv. (by H/W 7).
Applying Thm 6.46 to $T_W \in L(W)$, we obtain pairwise orthogonal T-inv. subspaces $W_1, \dots, W_k$ of W s.t. $W = W_1 \oplus \dots \oplus W_k$ and $1 \le \dim(W_i) \le 2$.
Each T_{W_i} is a rotation (if T_{W_i} is a reflection, $\exists\, x \neq 0$ in W_i s.t. $T(x) = -x$. But then $x \in W_i \cap E \subseteq$
$\subseteq E^\perp \cap E = \{0\}$, a contradiction).
If $E = \{0\}$ – (c) follows.
If $E \neq \{0\}$ – choose an orthonormal basis β for E containing p vectors, for some $p > 0$.
We can write β as a disjoint union $\beta = \beta_1 \cup \dots \cup \beta_r$ s.t.:
- each β_i contains exactly 2 vectors for $i < r$,
- β_r contains 2 vectors in p is even, and 1 vector if p is odd.

For each $i = 1, \dots, r$, let $W_{k+i} = \mathrm{Span}(\beta_i)$.
Then $W_1, \dots, W_{k+r}$ are pairwise orthogonal, and
$V = W \oplus E = W_1 \oplus \dots \oplus W_k \oplus \dots \oplus W_{k+r}$. $(*)$
Moreover, if any β_i contains 2 vectors, then $\det(T_{W_{k+i}}) = \det([T_{W_{k+i}}]_{\beta_i}) \underset{\text{as } T(x) = -x \text{ for all } x \in E}{=} \det\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = 1$.

So $T_{W_{k+i}}$ is a rotation, hence T_{W_j} is a rotation for $j < k+r$.

If β_r consists of 1 vector, then $\dim(W_{k+r}) = 1$ and $\det(T_{W_{k+r}}) = \det([T_{W_{k+r}}]_{\beta_r}) \overset{T(x)=-x \text{ on } E}{=} \det(-1) = -1$

Thus $T_{W_{k+r}}$ is a reflection by Example 1.

Hence the decomposition (*) satisfies (c).

Finally, we obtain the desired decomposition of a general orthogonal operator.

Cor Let V be a R.I.P.S., $\dim(V) < \infty$ and $T \in \mathcal{L}(V)$ is orthogonal.

Then there exist orthogonal operators $T_1, \dots, T_m$ on V such that:

a) For each i, T_i is either a reflection or a rotation.

b) For at most one i, T_i is a reflection.

c) $T_i T_j = T_j T_i$ for all i, j.

d) $T = T_1 T_2 \dots T_m$.

e) $\det(T) = \begin{cases} 1 & \text{if } T_i \text{ is a rotation for each } i, \\ -1 & \text{otherwise.} \end{cases}$

Proof

As in the proof of Thm 6.47(c), we can write

$V = W_1 \oplus \dots \oplus W_m$

where T_{W_i} is a rotation if $i < m$.

For each $i = 1, \dots, m$, define $T_i : V \to V$ by

$T_i(x_1 + \dots + x_m) = x_1 + \dots + x_{i-1} + T(x_i) + x_{i+1} + \dots + x_m$,

where $x_j \in W_j$ for all j.

Claim T_i is a reflection/rotation on V $\Leftrightarrow$ T_{W_i} is a rotation/reflection.

This claim is immediate from definition of reflection/rotation, with the subspace in the definition given by W_i.

This gives a) and b); (c), (d), (e) — exercise (H/W 7).