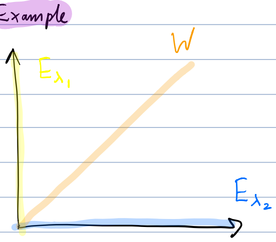
Thm Let $T \in \mathcal{L}(V)$, and assume that every subspace of V is T-invariant.
Then there is some scalar $c \in F$ s.t. $T(x) = c \cdot x$ for all $x \in V$.

Proof Let $x \in V$. Let $W_x = \mathrm{Span}(x)$, then $x \in W$ and W is T-inv. by assumption, so $T(x) \in W_x$.
That is, $T(x) = c_x \cdot x$ for some $c_x \in F$, for every $x \in W$.
So fix any $0 \neq x \in V$, and let c_x be as above. Consider the eigenspace E_{c_x}, $x \in E_{c_x}$.
Assume that $y \neq 0$ is any vector in V with $c_x \neq c_y$.
Then $T(x+y) = T(x) + T(y) \Rightarrow c_{x+y}(x+y) = c_x x + c_y \cdot y \Rightarrow (c_{x+y} - c_x)x + (c_{x+y} - c_y)y = 0$.
Both coefficients can't be 0, as then $c_x = c_{x+y} = c_y$ – contradicting the assumption.
So the set $\{x, y\}$ is lin. dep., hence $y \in \mathrm{Span}(x) \subseteq E_{c_x}$.
But this shows that $E_{c_x} = V$, hence $T(y) = c_x \cdot y$ for all $y \in V$.

Example



Let $V = \mathbb{R}^2$ and $T \in \mathcal{L}(V)$ defined by $T(a,b) = (\lambda_1 a, \lambda_2 b)$ for some $0 \neq \lambda_1 \neq \lambda_2 \in \mathbb{R}$.
Then T is diag'z, with eigenvalues $\lambda_1 \neq \lambda_2$ and
$E_{\lambda_1} = \{(a,0) : a \in \mathbb{R}\}$, $E_{\lambda_2} = \{(0,b) : b \in \mathbb{R}\}$, $\mathbb{R}^2 = E_{\lambda_1} \oplus E_{\lambda_2}$. Take any $c \in \mathbb{R}$.
Let W_c be the subspace $\{(a, ca) : a \in \mathbb{R}\}$.
Then W_c is T-invariant $\Leftrightarrow \forall a,\ T(a, ca) = (\lambda_1 a, \lambda_2 c a) = (b, cb)$ for some $b \in \mathbb{R}$.

$$\Downarrow$$

$\lambda_1 = b$, $\lambda_2 c = cb = \lambda_1$. (taking $a=1$)

Hence if $c \neq \frac{\lambda_1}{\lambda_2}$, then W_c is not T-inv.

Note: if $c \neq \frac{\lambda_1}{\lambda_2}$, then $W \cap E_{\lambda_1} = W \cap E_{\lambda_1} = \{0\}$, so $(W \cap E_{\lambda_1}) \cup (W \cap E_{\lambda_2}) \neq W$!