Dual spaces, continued

Def The lin. transformation T^t defined in Thm 2.25 is called the transpose of T.
It is the unique lin. transformation U s.t. $[U]_{\gamma^*}^{\beta^*} = ([T]_\beta^\gamma)^t$.

Ex 5 Define $T: P_1(\mathbb{R}) \to \mathbb{R}^2$ by $T(p(x)) = (p(0), p(2))$. This is a lin. transformation.
Let β and γ be the standard ordered bases for $P_1(\mathbb{R})$ and $\mathbb{R}^2$, respectively. We have $[T]_\beta^\gamma = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$.
We compute $[T^t]_{\gamma^*}^{\beta^*}$ from its definition.
Let $\beta^* = \{f_1, f_2\}$, $\gamma^* = \{g_1, g_2\}$, and suppose that $[T^t]_{\gamma^*}^{\beta^*} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
Then $T^t(g_1) = af_1 + cf_2$. So $T^t(g_1)(1) = (af_1 + cf_2)(1) = af_1(1) + cf_2(1) = a(1) + c(0) = a$.
But also $(T^t(g_1))(1) = g_1(T(1)) = g_1(1,1) = 1$. So $a = 1$.
Using similar computations (Exercise!) we obtain $c = 0$, $b = 1$ and $d = 2$. Hence we get
$[T^t]_{\gamma^*}^{\beta^*} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = ([T]_\beta^\gamma)^t$ – as predicted by Thm 2.25.

Next we consider the double dual space V^{**} of V.

Def. For a vector $x \in V$, we define $\hat{x}: V^* \to F$ by $\hat{x}(f) = f(x)$ for every $f \in V^*$.
Then $\hat{x}$ is a lin. functional on V^* $\left(\hat{x}(a \cdot f + g) = (a \cdot f + g)(x) = a \cdot (f(x)) + g(x) = a \cdot \hat{x}(f) + \hat{x}(g)\right)$, so $\hat{x} \in V^{**}$.

Lemma Let V be a fin. dim. v.s., and $x \in V$. If $\hat{x}(f) = 0$ for all $f \in V^*$, then $x = 0$.
Proof. Let $x \neq 0$. We show that there exists $f \in V^*$ s.t. $\hat{x}(f) \neq 0$.
Choose an ordered basis $\beta = \{x_1, \dots, x_n\}$ for V s.t. $x_1 = x$.
Let $\{f_1, \dots, f_n\}$ be the dual basis of β. Then $f_1(x_1) = 1 \neq 0$. Let $f = f_1$.

Thm 2.26 Let V be a fin. dim. v.s., and define the map $\psi: V \to V^{**}$ by $\psi(x) = \hat{x}$. Then ψ is an isomorphism.
Proof. a) ψ is linear:
Let $x, y \in V$ and $c \in F$. For $f \in V^*$, we have $\psi(cx+y)(f) = f(cx+y) = cf(x) + f(y) = c\hat{x}(f) + \hat{y}(f) = (c\hat{x} + \hat{y})(f)$.

Therefore $\psi(cx+y) = c\hat{x} + \hat{y} = c\psi(x) + \psi(y)$.
b) ψ is one-to-one: Suppose that $\psi(x)$ is the zero functional on V^* for some $x \in V$. Then $\hat{x}(f) = 0$ for every $f \in V^*$.
By the previous lemma, $x = 0$. So $N(\psi) = \{0\}$.
c) ψ is an isomorphism This follows from (b) and the fact that $\dim(V) = \dim(V^{**})$.

Corollary. Let V be a fin. dim. v.s. with dual space V^*. Then every ordered basis for V^* is the dual basis for some basis for V.
Proof. Let $\{f_1, \dots, f_n\}$ be an ordered basis for V^*. Combining Thm 2.24 and Thm 2.26, there exists a dual basis $\{\hat{x}_1, \dots, \hat{x}_n\}$ in V^{**}, that is $\hat{x}_i(f_j) = f_j(x_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ for all i and j.
Thus $\{f_1, \dots, f_n\}$ is the dual basis of $\{x_1, \dots, x_n\}$.

115A Reminder: Eigenvectors

Def A matrix $A \in M_{n \times n}(F)$ is diagonal if $A_{ij} = 0$ for all $i \neq j$.

Def A lin. operator $T \in \mathcal{L}(V)$, $\dim(V) < \infty$, is diagonalizable if there exists an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

Def 1) Given $A, B \in M_{n \times n}(F)$, we say that A and B are similar if there exists an invertible matrix $Q \in M_{n \times n}(F)$ such that $B = Q^{-1}AQ$.
2) A matrix $A \in M_{n \times n}(F)$ is diagonalizable if A is similar to a diagonal matrix.

Thm Let $T \in \mathcal{L}(V)$, $\dim(V) < \infty$ and β, γ ordered bases for V.
Then $\det([T]_\beta) = \det([T]_\gamma)$.
Thus, $\det([T]_\beta)$ does not depend on β, and is called the determinant of T, or $\det(T)$.

Proposition

1) T is bijective $\Leftrightarrow$ $\det T \neq 0$.
2) T is bijective $\Rightarrow$ $\det(T^{-1}) = (\det T)^{-1}$.
3) If $U \in \mathcal{L}(V)$, then $\det(TU) = \det(T) \cdot \det(U)$.

Thm Let $T \in \mathcal{L}(V)$, $\dim(V) < \infty$ and β an ord. basis for V. Then:
T is diagonalizable $\Leftrightarrow$ $[T]_\beta$ is diagonalizable.

Corollary $A \in M_{n \times n}(F)$ is diagonalizable $\Leftrightarrow$ L_A is diagonalizable.

Def 1) Let $T \in \mathcal{L}(V)$, $\dim(V) < \infty$.
A non-zero vector $v \in V$ is an eigenvector of T if $T(v) = \lambda v$ for some $\lambda \in F$.
Then λ is an eigenvalue of T corresponding to the eigenvector v.
2) Let $A \in M_{n \times n}(F)$.
A non-zero $v \in F^n$ is an eigenvector of A if $Av = \lambda v$ for some $\lambda \in F$.
Then λ is an eigenvalue of A corresp. to v.

Thm A lin. operator $T \in \mathcal{L}(V)$ is diagonalizable if and only if there exists an ordered basis β for V consisting of eigenvectors for T.
Furthermore, then
$$[T]_\beta = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix},$$
where λ_i is the eigenvalue corresponding to the i^{th} vector in β.

Thm 5.2 $\lambda \in F$ is an eigenvalue of T $\Leftrightarrow$ $\det(T - \lambda I_V) = 0$.

Corollary For $A \in M_{n \times n}(F)$, $\lambda \in F$ is an eigenvalue of A $\Leftrightarrow$ $\det(A - \lambda I_n) = 0$.

Def 1) For $A \in M_{n \times n}(F)$, the polynomial $f(t) = \det(A - t I_n)$ is called the characteristic polynomial of A.
2) Let V be a v.s., $\dim(V) = n$ and β is an ord. basis for V. Let $T \in \mathcal{L}(V)$
We define the characteristic polynomial of T to be $f(t) = \det([T]_\beta - t I_n)$

Properties of char. polynomials

Let $A \in M_{n \times n}(F)$ be given, let $f(t)$ be its char. polynomial.
1) $f(t)$ is of degree n, and moreover $f(t) = (-1)^n t^n + c_{n-1} t^{n-1} + \ldots + c_0$ for some $c_0, \ldots, c_{n-1} \in F$.

2) $\lambda \in F$ is an eigenvalue of A $\Leftrightarrow$ $f(\lambda) = 0$ (that is, λ is a root of $f(t)$).
3) A has at most n distinct eigenvalues (as $f(t)$ has at most n roots).
4) If $\lambda \in F$ is an eigenvalue of A, then:
$x \in F^n$ is an eigenvector of A corresp. to λ $\Leftrightarrow$ $x \neq 0$ and $x \in N(L_A - \lambda I_{F^n})$.

Theorem 5.5

Let $T \in \mathcal{L}(V)$ be a lin. operator on V, and let $\lambda_1, \ldots, \lambda_k$ be distinct eigenvalues of T. If $v_1, \ldots, v_k$ are e.vect's of T s.t. v_i corresponds to λ_i, then the set
$\{v_1, \ldots, v_k\}$
is lin. indep.

Corollary Let $T \in \mathcal{L}(V)$ and $\dim(V) = n$.
If T has n distinct e.val's, then T is diagonalizable.

Def A polynomial $f(t) \in P(F)$ splits over F if there are scalars $c, a_1, \ldots, a_n \in F$ (not necessarily distinct) such that
$f(t) = c(t-a_1)(t-a_2) \cdot \ldots \cdot (t-a_n)$.

Thm 5.6 The char. polynomial of any diagonalizable lin. operator splits.

Def Let λ be an e.val. of a lin. operator or matrix with char. polynomial $f(t)$. The (algebraic) multiplicity of λ is the largest positive integer k for which $(t-\lambda)^k$ is a factor of $f(t)$. (That is, $f(t)$ can be written as $f(t) = (t-\lambda)^k g(t)$ for some polynomial $g(t)$).

Def Let $T \in \mathcal{L}(V)$, λ an eigenvalue of T. We define E_λ, the eigenspace of T corresp. to λ, as

$E_\lambda = \{ x \in V : T(x) = \lambda x \} = N(T - \lambda I_V)$. (and similary for a matrix).

Note that this is a subspace of V, consisting of 0 and the e.vects of T corresp to λ.

Thm 5.7. Let $T \in \mathcal{L}(V)$, $\dim(V) < \infty$, λ an e.val. of T with multiplicity m.
Then $1 \leq \dim(E_\lambda) \leq m$.

Invariant and cyclic subspaces

Def. Let $T : V \to V$ be a lin. operator on V. A subspace W of V is called a T-invariant subspace of V if $T(W) \subseteq W$, that is, if $T(v) \in W$ for all $v \in W$.

Ex1 Let T be any lin. operator on V. Then the following subspaces of V are T-invariant:

1) $\{0\}$ (as $T(0) = 0$),
2) V,
3) $R(T)$,
4) $N(T)$, ($v \in N(T) \Leftrightarrow T(v) = 0$, then if $w = T(v)$, we have $T(w) = T(\underbrace{T(v)}_{=0}) = T(0) = 0$, so $w \in N(T)$).
5) E_λ, for any eigenvalue λ of T. (if v is an eigen vect., by def. $T(v) = \lambda v$ for some $\lambda \in F$, so $T(v) \in \text{Span}(v)$. Hence $T(E_\lambda) \subseteq \text{Span}(E_\lambda) = E_\lambda$ as E_λ is a subspace of V).

Ex 2. Let T be a lin. op. on $V = \mathbb{R}^3$ defined by
$T(a,b,c) = (a+b, b+c, 0)$.

Then: 1| The xy-plane $W_1 = \{(x,y,0) : x, y \in \mathbb{R}\}$ is a T-inv. subspace.
($v \in W_1 \Rightarrow v = (a,b,0)$ for some $a,b \in \mathbb{R} \Rightarrow T(v) = (a+b, b, 0) \in W_1$).
2) The x-axis $W_2 = \{(x,0,0) : x \in \mathbb{R}\}$ is a T-inv. subspace
($v \in W_2 \Rightarrow v = (a,0,0)$ for some $a \in \mathbb{R} \Rightarrow T(v) = (a+0, 0+0, 0) = (a,0,0) \in W_1$).

Def. Let T be a lin. op. on V. And let $x \in V$ be a non-zero vector. The subspace
$W = \text{Span}(\{x, T(x), T^2(x), \ldots\})$ is called the T-cyclic subspace of V generated by x.

Rem It is the "smallest" T-invariant subspace of V containing x. That is, any T-invariant subspace of V containing x must also contain W. Indeed, if $Z \subseteq V$ is a T-inv. subspace of V and $x \in Z$, then by T-inv. also $T(x) \in Z$, $T^2(x) = T(T(x)) \in Z$, … . So $\{x, T(x), T^2(x), \ldots\} \subseteq Z$, and as Z is a subspace we get $W = \text{Span}(\{x, T(x), T^2(x), \ldots\}) \subseteq Z$.

Ex 3 Let T be a lin. op. on $\mathbb{R}^3$ defined by $T(a,b,c) = (-b+c,\ a+c,\ 3c)$.
We determine the T-cyclic subspace generated by $e_1 = (1,0,0)$.
$T(e_1) = T(1,0,0) = (0,1,0) = e_2$
$T^2(e_1) = T(T(e_1)) = T(e_2) = (-1,0,0) = -e_1$.
Then $T^3(e_1) = T(T^2(e_1)) = T(-e_1) = -T(e_1) = -e_2$, $T^4(e_1) = T(T^3(e_1)) = T(-e_2) = -T(e_2) = e_1$.
Hence $T^n(e_1) \in \{e_2, -e_1, -e_2, e_1\}$ for all n, hence $\mathrm{Span}(\{e_1, T(e_1), T^2(e_1), \dots\}) = \mathrm{Span}(\{e_1, e_2\}) = \{(s,t,0) : s,t \in \mathbb{R}\}$.

Def If T is a lin. op. on V and W is a T-invariant subspace of V, we consider the map $T_W : W \to W$, the restriction of T to W, defined by $T_W(x) = T(x)$ for all $x \in W$ (note that $T(x) \in W$ as W is T-inv.).
Then T_W is a lin. op. on W. ($\forall x,y \in W, a \in F$ we have $T_W(ax+y) = T(ax+y) \overset{T\text{ lin.}}{=} aT(x) + T(y) \overset{\text{def}}{=} aT_W(x) + T_W(y)$).

As a lin. op., T_W inherits certain properties from its parent operator T.

Thm 5.21 Let T be a lin. op. on a fin. dim. v.s. V, and let W be a T-inv. subspace of V.
Then the characteristic polynomial of T_W divides the characteristic polynomial of T.

Proof. Choose an ordered basis $\gamma = \{v_1, v_2, \dots, v_k\}$ for W, and extend it to an ordered basis $\beta = \{v_1, v_2, \dots, v_k, v_{k+1}, \dots, v_n\}$ for V.
Let $A = [T]_\beta$ and $B_1 = [T_W]_\gamma$. Then we have

$$A = \begin{pmatrix} | & | & & | & | & & | \\ [T(v_1)]_\beta & [T(v_2)]_\beta & \cdots & [T(v_k)]_\beta & [T(v_{k+1})]_\beta & \cdots & [T(v_n)]_\beta \\ | & | & & | & | & & | \end{pmatrix}.$$

As W is T-inv, we have that $T(v_i) \in W$ for $1 \le i \le k$. Then by the choice of β and γ, for every vector $x \in W$ we have $[x]_\beta = \begin{pmatrix} [x]_\gamma \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \}k \\ \}n-k \end{matrix}$, in particular $[T(v_i)]_\beta = \begin{pmatrix} [T(v_i)]_\gamma \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \}k \\ \}n-k \end{matrix}$ for $1 \le i \le k$, so $A = \begin{pmatrix} \underbrace{\begin{matrix} B_1 \\ 0 \end{matrix}}_{k} & \underbrace{\begin{matrix} B_2 \\ B_3 \end{matrix}}_{n-k} \end{pmatrix} \begin{matrix} \}k \\ \}n-k \end{matrix}$ for some $B_2 \in M_{k\times k}(F)$, $B_3 \in M_{(n-k)\times(n-k)}(F)$.

Let $f(t)$ be the char. polynomial of T and $g(t)$ the char. polynomial of T_W. Then we have:

$$f(t) \overset{\text{def}}{=} \det(A - tI_n) = \det\begin{pmatrix} B_1 - tI_k & B_2 \\ 0 & B_3 - tI_{n-k} \end{pmatrix} = g(t) \cdot \det(B_3 - tI_{n-k}).$$ Thus $g(t)$ divides $f(t)$.

Ex 5 Let T be the lin. op. on $V = \mathbb{R}^4$ defined by
$T(a,b,c,d) = (a+b+2c-d,\ b+d,\ 2c-d,\ c+d)$, and let $W = \{(t,s,0,0) : t,s \in \mathbb{R}\}$.
- W is T-inv: for any $(a,b,0,0) \in W$ we have $T(a,b,0,0) = (a+b,\ b,\ 0,0) \in W$.
- Let $\gamma = \{e_1, e_2\}$, it is an ordered basis for W. Extend it to the standard basis $\beta = \{e_1, e_2, e_3, e_4\}$ for $\mathbb{R}^4$. Then:

$$B_1 = [T_W]_\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A = [T]_\beta = \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

- $$\overset{\text{char. poly of } T}{f(t)} = \det(A - tI_4) = \det\begin{pmatrix} 1-t & 1 & 2 & -1 \\ 0 & 1-t & 0 & 1 \\ 0 & 0 & 2-t & -1 \\ 0 & 0 & 1 & 1-t \end{pmatrix} = \det\begin{pmatrix} 1-t & 1 \\ 0 & 1-t \end{pmatrix} \cdot \det\begin{pmatrix} 2-t & -1 \\ 1 & 1-t \end{pmatrix} = \overset{\text{char. poly. of } T_W}{g(t)} \cdot \det\begin{pmatrix} 2t & -1 \\ 1 & 1-t \end{pmatrix}.$$

In view of Thm 5.21, we may use the char. poly. of T_W to gain information about the char. poly. of T itself.
For this cyclic subspaces are useful because for them we can obtain an explicit formula for char. poly. as follows:

Thm 5.22
Let T be a lin. op. on V, $\dim(V) < \infty$. Let W be the T-cyclic subspace of V generated by $v \in V$, $v \neq 0$. Let $k = \dim(W)$.
a) $\{v, T(v), T^2(v), \dots, T^{k-1}(v)\}$ is a basis for W.
b) If $a_0 v + a_1 T(v) + \dots + a_{k-1} T^{k-1}(v) + T^k(v) = 0$, then the char. poly. of T_W is
$f(t) = (-1)^k (a_0 + a_1 t + \dots + a_{k-1} t^{k-1} + t^k)$.

Proof.

a) Since $v \neq 0$, the set $\{v\}$ is lin. indep.

Let j be the largest positive integer for which $\beta = \{v, T(v), \ldots, T^{j-1}(v)\}$ is lin. indep.

(such a j must exist because $\dim(V) < \infty$).

Let $Z = \operatorname{Span}(\beta)$. Then β is a basis for Z. Furthermore, $T^j(v) \in Z$.

Claim. Z is a T-inv. subspace of V.

Let $w \in Z$. Since w is a lin. combination of the vectors of β, $\exists\, b_0, b_1, \ldots, b_{j-1} \in F$ s.t.

$w = b_0 v + b_1 T(v) + \cdots + b_{j-1} T^{j-1}(v)$, hence using linearity of T:

$T(w) = b_0 T(v) + b_1 T^2(v) + \ldots + b_{j-1} T^j(v)$.

Thus $T(v)$ is a lin. comb. of vectors in Z, hence $T(w) \in Z$ – so Z is T-inv.

We have $v \in Z$. As the cyclic space W is the smallest T-inv. subspace containing v, we must have $W \subseteq Z$.

As $\beta \subseteq W$, we also have $Z = \operatorname{Span}(\beta) \leq W$. So $Z = W$.

It follows that β is a basis for W, hence $\dim(W) = j$. Thus $j = k$, and this proves a).

b) Now view β from a) as an ordered basis for W. Let $a_0, \ldots, a_{k-1} \in F$ be s.t.

$(*)\ a_0 v + a_1 T(v) + \ldots + a_{k-1} T^{k-1}(v) + T^k(v) = 0$. We calculate $[T]_\beta$. For each vector in the basis β we have:

$$[T(v)]_\beta = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad [T(T(v))]_\beta = [T^2(v)]_\beta = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \ldots, [T(T^{k-2}(v))]_\beta = [T^{k-1}(v)]_\beta = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \ [T(T^{k-1}(v))]_\beta = [T^k(v)]_\beta \underset{\underset{\text{using } (*)}{\uparrow}}{=} \begin{pmatrix} -a_0 \\ -a_1 \\ \vdots \\ -a_{k-1} \end{pmatrix}.$$

Hence $[T_W]_\beta = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix}$. By induction on k one can shot that the char. poly is equal to $f(t) = (-1)^k (a_0 + a_1 t + \ldots + a_{k-1} t^{k-1} + t^k)$. (Homework!)

Thus $f(t)$ is the char. poly of T_W, proving b).