115 A Reminder Chapter I + II

Vector spaces

Definition. A vector space V over a field F is a set with two operations, addition and scalar multiplication, (so for any x, y in V and $a \in F$, $x+y$ and ax are in V) such that the following conditions hold.

(VS1) $x+y = y+x$ for all x, y in V (commutativity)

(VS2) $(x+y)+z = x+(y+z)$ for all x, y, z in V (associativity)

(VS3) There exists an element 0 in V such that $x+0 = x$ for all $x \in V$. (identity)

(VS4) For each x in V there is an element y in V such that $x+y=0$ (y is an inverse of x)

(VS5) $1 \cdot x = x$ for all x in V (where 1 is the multiplicative identity of F).

(VS6) $a(bx) = (ab)x$ for all x in V and a, b in F

(VS7) $a(x+y) = ax+ay$ for all a in F and x, y in V. } distributive laws

(VS8) $(a+b)x = ax+bx$ for all a, b in F and x in V.

Elements of V are called vectors.

Elements of F are called scalars.

Theorem 1.1 (cancellation law)

Let V be a v.s. and let $x, y, z \in V$.

If $x+z = y+z$, then $x = y$.

Corollary.

1) In any vector space V, there is a unique element 0 satisfying (VS3) — the zero vector of V.

2) For any v.s. V and any x in V, there is a unique element y in V satisfying (VS4). It is called the inverse of x, and denoted by $-x$.

Theorem 1.2. Let V be a v.s. over F.

For all x in V and a in F we have:

1) $0 \cdot x = 0$ (Note: the 1st 0 is a scalar in F, the 2nd one is the zero vector in V).

2) $(-a) \cdot x = -(ax) = a(-x)$

3) $a \cdot 0 = 0$ (Note: this is the zero vector of V on both sides).

Subspaces.

Definition. Let V be a v.s. A subset $W \subseteq V$ is a subspace of V if W itself is a v.s. with respect to the addition and scalar multiplication defined on V.

Theorem 1.3. Let V be a v.s., and let $W \subseteq V$ be a subset of V.

Then W is a subspace of V if and only if all of the following conditions hold:

(a) $0 \in W$

(b) $x+y \in W$ for all $x, y \in W$ (W is closed under addition)

(c) $c \cdot x \in W$ for all $c \in F$ and $x \in W$ (W is closed under scalar multiplication).

Theorem 1.4. Let V be a v.s. over F.

If $W_1, \dots, W_n$ are subspaces of V, then the set $W = W_1 \cap W_2 \cap \dots \cap W_n$ is also a subspace of V.

Linear combinations

Definition. Let V be a v.s., and let $S \subseteq V$ be a non-empty subset of V.

1) A vector v in V is a linear combination of S if one can write

$v = a_1 u_1 + a_2 u_2 + \dots + a_n u_n$

for some vectors $u_1, \dots, u_n$ in S and some scalars $a_1, \dots, a_n$ in F.

2) The span of S, denoted $\text{Span}(S)$, is the subset of V consisting precisely of all linear combinations of S. That is,

$\text{Span}(S) = \{a_1u_1 + \dots + a_nu_n : n \in \mathbb{N},\ a_i \in F,\ u_i \in S\}$.

For convenience, we define $\text{Span}(\emptyset) = \{0\}$.

Theorem 1.5. Let S be any subset of a v.s. V. Then:

1) $\text{Span}(S)$ is a subspace of V.
2) Any subspace of V that contains S must also contain $\text{Span}(S)$.

Definition. Let V be a v.s. and S a subset of V.
We say that S generates (or spans) V if $\text{Span}(S) = V$.

Definition. A subset S of a v.s. V is linearly dependent if there exist a finite number of distinct vectors $u_1, \dots, u_n$ in S and scalars $a_1, \dots, a_n \in F$, with at least one $a_i \neq 0$, such that

$a_1u_1 + \dots + a_nu_n = 0$.

We call S linearly independent if it is not linearly dependent.

Theorem 1.6. Let V be a v.s. and $S_1 \subseteq S_2 \subseteq V$ be two subsets of V.

1) If S_1 is lin. dependent, then S_2 is also linearly dependent.
2) If S_2 is lin. indep., then S_1 is also lin. indep.

Theorem 1.7. Let S be a lin. indep. subset of a vector space V.
Let v be any vector in V not contained in S.
Then $S \cup \{v\}$ is lin. dep. if and only if $v \in \text{Span}(S)$.

Bases and dimension.

Definition. A basis for a v.s. V is a subset of V which is lin. indep. and generates V.

Theorem 1.8. A subset $\{u_1, \dots, u_n\}$ of a v.s. V is a basis if and only if every vector $v \in V$ can be written uniquely in the form

$v = a_1u_1 + \dots + a_nu_n$,

where $a_i \in F$.

(so "uniquely" here means that there is only one possible choice of the scalars $a_1, \dots, a_n \in F$ satisfying the equality)

Theorem 1.9. If a v.s. V is generated by a finite subset S, then some subset of S is a basis for V.
It follows that every finitely generated v.s. has a basis.

Theorem 1.10. (Replacement theorem)

Let V be a v.s. generated by a set $G \subseteq V$ with $|G| = n$, and let L be a lin. indep. subset of V, $|L| = m$.
Then $m \leq n$, and there exists $H \subseteq G$ with $|H| = n - m$ such that $L \cup H$ generates V.

Corollary 1. Let V be a finitely generated v.s. Then every basis for V has the same number of elements.

Definition. A v.s. V is finite-dimensional if it has a finite basis.
The (unique) number of vectors in a basis for V is called the dimension of V, denoted $\dim(V)$.
If there is no finite basis, then V is infinite-dimensional.

Corollary 2. Let V be a v.s. of dimension n. Then:

a) Any generating set for V must contain at least n vectors.
b) Any lin. indep. subset of V with n elements is a basis.
c) Every lin. indep. subset of V can be extended to a basis for V.

Theorem 1.11. Let W be a subspace of a v.s. V with $\dim(V) < \infty$.
Then $\dim(W) \leq \dim(V)$.
Moreover, if $\dim(W) = \dim(V)$, then $V = W$.

Linear transformations

Def Let V and W be v.s. over the same field of scalars F.
A lin. transformation from V to W is a function $T: V \to W$ satisfying
1) $T(x+y) = T(x) + T(y)$ for all $x, y \in V$.
2) $T(cx) = cT(x)$ for all $x \in V$ and $c \in F$.

Properties of lin. transformations
1) Let $T: V \to W$ be a lin. transf. Then:
a) $T(0) = 0$
b) $T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i)$ for all $x_i \in V$, $a_i \in F$.

2) A function $T: V \to W$ is a lin. transf. $\iff$ $T(cx+y) = cT(x) + T(y)$ for all $x, y \in V$, $c \in F$.

Thm 2.6 Let V, W be v.s. v.s. over a field F, and let $\{v_1, \dots, v_n\}$ be a basis for V.
Then for any vectors $w_1, \dots, w_n \in W$ there exists exactly one lin. transf. $T: V \to W$ s.t.
$T(v_i) = w_i$ for $1 \leq i \leq n$.

Def Let $T: V \to W$ be a lin. transf.
1) T is injective if $T(v) = T(u)$ imlies $v = u$, for all $u, v \in V$.
2) T is surjective if for every $w \in W$ there is some $v \in V$ s.t. $T(v) = w$.
3) T is bijective if it is both injective and surjective.

Null space and range

Def Let V, W be v.s. and $T: V \to W$ a lin. transf.
1) The null space of T is defined as
$N(T) = \{x \in V : T(x) = 0\}$.
2) The range of T is the image of V under T, that is the set
$R(T) = \{y \in W : y = T(x) \text{ for some } x \in V\}$.

Thm 2.1
1) $N(T)$ is a subspace of V.
2) $R(T)$ is a subspace of W.

Thm 2.4 Let $T: V \to W$ be a lin. transf.
1) T is injective $\iff$ $N(T) = \{0\}$.
2) T is surjective $\iff$ $R(T) = W$.

Thm 2.2 Let $T: V \to W$ be a lin. transf.
Assume $\beta = \{v_1, \dots, v_n\}$ is a basis for V.
Then $R(T) = \operatorname{Span}(\{T(v_1), \dots, T(v_n)\})$.

Thm 2.3 (Dimension Theorem)
Let V, W be v.s., $T: V \to W$ a lin. transf., and $\dim(V) < \infty$. Then
$\dim(V) = \dim(N(T)) + \dim(R(T))$.

Thm 2.5 Let $T: V \to W$ be a lin. transf., and assume $\dim(V) = \dim(W)$.
Then the following are equivalent:

1) T is injective.
2) T is surjective.
3) T is bijective.
4) $\dim(R(T)) = \dim(V)$.

The vector space of linear transformations $\mathcal{L}(V,W)$.

Def. Let V, W be v.s. over F, and let $T, U : V \to W$ be linear transformations.
Then we define the functions $T+U$ and aT, for every $a \in F$, by :

$(T+U)(x) = T(x) + U(x)$ for all $x \in V$.

$(aT)(x) = a \cdot T(x)$ for all $x \in V$.

Thm 2.7
If T and U are linear, then $T+U$ and aU are also linear.

Def We denote the set of all lin. transf.'s from V to W by $\mathcal{L}(V,W)$.
Then it is a v.s. over F, with the operations of addition and scalar multiplication described above.
When $W = V$, we write $\mathcal{L}(V)$ instead of $\mathcal{L}(V,V)$.

Matrix representation of a lin. transf.

Def Let V be a v.s. with $\dim(V) < \infty$. An ordered basis for V is a basis for V with a specified order on its vectors.

Def Let $\beta = \{v_1, \dots, v_n\}$ be an ordered basis for V.
Then any vector $x \in V$ can be written as

$x = a_1 v_1 + \dots + a_n v_n$ for some unique scalars $a_1, \dots, a_n \in F$.

We define the coordinate vector of x relative to β by

$$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in F^n.$$

Def Let V, W be v.s. with ordered bases $\beta = \{v_1, \dots, v_n\}$ and $\gamma = \{w_1, \dots, w_m\}$, respectively.
Let $T: V \to W$ be a lin. transformation.
Then the matrix representation of T in the ordered bases β and γ is defined as the matrix $[T]_\beta^\gamma \in M_{m \times n}(F)$ given by

$$[T]_\beta^\gamma = \begin{pmatrix} | & | & & | \\ [T(v_1)]_\gamma & [T(v_2)]_\gamma & \cdots & [T(v_n)]_\gamma \\ | & | & & | \end{pmatrix},$$

where $[T(v_i)]_\gamma$ are the coordinates of the vector $T(v_i) \in W$ with respect to the basis γ.
If $V = W$ and $\beta = \gamma$, we simply write $[T]_\beta$.

Thm 2.8
Let V, W be fin. dim. v.s.'s with ordered bases β and γ, resp.
Let $T, U : V \to W$ be lin. transformations. Then:

1) $U = T$ (meaning that $U(x) = T(x)$ for all $x \in V$) $\iff [U]_\beta^\gamma = [T]_\beta^\gamma$.
2) $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$.
3) $[aT]_\beta^\gamma = a \cdot [T]_\beta^\gamma$ for all $a \in F$.

Composition of lin. transf's and matrix multiplication

Def Let V, W, Z be v.s.'s over F. Let $T: V \to W$ and $U: W \to Z$ be lin. transf's. Their composition is the function UT, from V to Z, defined by
$(UT)(x) = U(T(x))$ for all $x \in V$.

Thm 2.9 If T and U are linear, then UT is also linear.

Def Given matrices $A \in M_{m \times n}(F)$ and $B \in M_{n \times p}(F)$, we define the product $AB \in M_{m \times p}(F)$ to be the matrix with the entries

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} \cdot B_{kj} \quad , \text{ for } 1 \le i \le m, \ 1 \le j \le p.$$

Thm 2.11 Let V, W, Z be fin. dim. v.s.'s with ordered bases α, β, γ respectively. Let $T: V \to W$ and $U: W \to Z$ be lin. transformations. Then
$[UT]_\alpha^\gamma = [U]_\beta^\gamma \cdot [T]_\alpha^\beta$.

Corollary Let V be a fin. dim. v.s. with an ordered basis β.
Let $T, U \in \mathcal{L}(V)$.
Then $[UT]_\beta = [U]_\beta [T]_\beta$.

Thm 2.14 Let V, W be fin. dim. v.s.'s with ordered bases β and γ, resp.
Let $T: V \to W$ be a lin. transf. Then for each vector $u \in V$ we have:
$[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta$.
(so, we calculate the coordinates of the vector $T(u)$ from the coordinates of the vector u).

Def To every matrix $A \in M_{m \times n}(F)$, we associate a linear transformation $L_A: F^n \to F^m$ defined by
$L_A(x) = Ax$ for every (column) vector $x \in F^n$.
We call L_A the left-multiplication transformation

Invertibility

Def Let V, W be v.s.'s and $T: V \to W$ linear.
1) A lin. transf. $U: W \to V$ is the inverse of T if
$UT = I_V$ and $TU = I_W$.
2) T is invertible if it has an inverse.

Basic facts
1) If T is invertible, then its inverse is <u>unique</u>, and is denoted by T^{-1}.
2) T is invertible $\iff$ T is a bijection.
3) If T, U are invertible, then
- $(TU)^{-1} = U^{-1} T^{-1}$
- $(T^{-1})^{-1} = T$.

Lemma Let $T: V \to W$ be lin. and invertible, and $\dim(V) < \infty$.
Then $\dim(V) = \dim(W)$.

Def A matrix $A \in M_{n \times n}(F)$ is invertible if there exist $B \in M_{n \times n}(F)$ s.t. $AB = BA = I$.
If such a B exists, the it is <u>unique</u>, called the inverse of A and denoted by A^{-1}.

Thm 2.18 Let V, W be fin. dim. v.s.'s with ordered bases β and γ, resp.
Let $T: V \to W$ be lin.
Then T is invertible $\Leftrightarrow$ the matrix $[T]_\beta^\gamma$ is invertible.
Furthermore, $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$.

Isomorphisms

Def Two v.s.'s V and W are isomorphic if there exists an invertible lin. transf. $T: V \to W$. Such a T is called an isomorphism from V onto W.

Thm 2.19 Two fin. dim. v.s.'s V and W are isomorphic $\Leftrightarrow$ $\dim(V) = \dim(W)$.

Corollary. Let V be a v.s. over F. Then V is isomorphic to F^n if and only if $\dim(V) = n$.

Thm 2.20 Let V, W be v.s.'s over F, $\dim(V) = n$, $\dim(W) = m$.
Let β, γ be ordered bases for V, W, resp.
Then the map $\phi: \mathcal{L}(V,W) \to M_{m\times n}(F)$ defined by
$$\phi(T) = [T]_\beta^\gamma \quad \text{for all } T \in \mathcal{L}(V,W)$$
is an isomorphism.

Corollary If $\dim(V) = n$, $\dim(W) = m$ then $\dim(\mathcal{L}(V,W)) = \dim(M_{m\times n}(F)) = mn$.

Dual spaces

Definition Let V be a vector space over a field of scalars F (which is itself a vector space of dim 1 over F.)
A linear transformation from V to F is called a linear functional on V.

Ex 1 Let V be the v.s. over $F=\mathbb{R}$ of continuous real-valued functions on the interval $[0;2\pi]$. Fix a function $g \in V$. Then the function $h: V \to \mathbb{R}$ defined by
$$h(x) = \frac{1}{2\pi}\int_0^{2\pi} x(t)\,g(t)\,dt \quad \text{for} \quad x \in V$$
is a lin. functional on V.
} — Definite integral is one of the most important lin. functionals!

Ex 2 Let $V = M_{n\times n}(F)$, and define $f: V \to F$ by $f(A) = tr(A)$, the trace of the matrix $A \in V$.
(recall: $tr(A) = A_{11} + A_{22} + \dots + A_{nn}$). Then f is a lin. functional.

Ex 3 Let V be a finite dim. v.s., and let $\beta = \{x_1, \dots, x_n\}$ be an ordered basis for V. For each $i = 1, 2, \dots, n$ we define $f_i(x) = a_i$, where $[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ is the coordinate vector of x relative to β. Then each f_i is a lin. functional on V called the i^{th} coordinate function w.r.t. the basis β.

Def For a v.s. V over F, we define the dual space of V to be the v.s. $\mathcal{L}(V,F)$, denoted by V^*. We define the double dual V^{**} of V to be the dual of V^*.

Thus V^* is the v.s. consisting of all lin. functionals on V.

Remark If V is finite dimensional, then by Theorem 2.20 we have:
$\dim(V^*) = \dim(\mathcal{L}(V,F)) = \dim(M_{\dim(V)\times\dim(F)}(F)) = \dim(V)\cdot\dim(F) = \dim(V)\cdot 1 = \dim(V)$.
Hence, by Thm 2.19, V and V^* are isomorphic. (this can be false when V is infinite dimensional!)

Thm 2.24 Supp. that V is a fin. dim. v.s. over F with the ordered basis $\beta = \{x_1, \dots, x_n\}$. Let f_i, $1 \le i \le n$ be the i^{th} coordinate function w.r.t. β (as in Example 3 above). Let $\beta^* = \{f_1, \dots, f_n\}$.
Then β^* is an ordered basis for V^*, and for any $f \in V^*$ we have $f = \sum_{i=1}^n f(x_i) f_i$.

Proof. Let $f \in V^*$ be arbitrary. Since $\dim(V^*) = n$ by the remark above, we only need to show that $f = \sum_{i=1}^n f(x_i) f_i$ (as this means that β^* generates V^*, and is of size n, hence is a basis for V^* by the Replacement Theorem).
Let $g = \sum_{i=1}^n f(x_i) f_i$. Then for $1 \le j \le n$ we have: $g(x_j) = \left(\sum_{i=1}^n f(x_i) f_i\right)(x_j) = \sum_{i=1}^n f(x_i) f_i(x_j)$.
Since by definition, $f_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$, we get $g(x_j) = f(x_j)$.

Hence f and g are two lin. transformations that agree on every vector in a basis, hence they are equal on the whole space (by Corollary to Thm 2.6).

Def Using the notation of Thm 2.24, we call the ordered basis $\beta^* = \{f_1, \dots, f_n\}$ of V^* that satisfies $f_i(x_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \ne j \end{cases}$ the *dual basis of β*.

Ex 4 Let $\beta = \{(2,1), (3,1)\}$ be an ordered basis for $V = \mathbb{R}^2$. Suppose that the dual basis of β is given by $\beta^* = \{f_1, f_2\}$. To determine a formula for f_1, we know that by def. it must satisfy the equations:
$1 = f_1(2,1) = f_1(2e_1 + e_2) = 2f_1(e_1) + f_1(e_2)$,
$0 = f_1(3,1) = f_1(3e_1 + e_2) = 3f_1(e_1) + f_1(e_2)$.
Solving the equations, we obtain $f_1(e_1) = -1$ and $f_1(e_2) = 3$. Hence $f_1(x,y) = -x + 3y$.
Similarly, we get $f_2(x,y) = x - 2y$.

Assume now that V, W are fin. dim. v.s. over F with ordered bases β and γ, respectively, $\dim(V) = m$, $\dim(W) = n$.
Recall (Section 2.4): there exists a one-to-one correspondence between lin. transformations $T: V \to W$ and $m \times n$ matrices over F given by $T \leftrightarrow [T]_\beta^\gamma$.

Question: Given a matrix $A = [T]_\beta^\gamma$, when is it possible to find a lin. transformation U represented by the matrix A^t in some basis?

Of course, if $m \ne n$ then it is impossible for U to be a lin. transformation from V to W. Dual spaces help!

Thm 2.25 Let V and W be fin. dim. v.s. over F with ordered bases β and γ, resp.
For any lin. transformation $T: V \to W$, the map $T^t: W^* \to V^*$ defined by
$T^t(g) = gT$ for all $g \in W^*$
is a lin. transformation such that $[T^t]_{\gamma^*}^{\beta^*} = \left([T]_\beta^\gamma\right)^t$.

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ | & & | \\ V^* & \xleftarrow{T^t} & W^* \end{array}$$

Proof.
1) For any $g \in W^*$, we defined $T^t(g) = gT$, i.e. the composition of linear maps $T: V \to W$ and $g: W \to F$.
Hence $T^t(g)$ is a linear map from $V \to F$, so it is an element of V^*. Thus indeed T^t maps W^* into V^*. Given any $g, h \in W^*$ and $a \in F$, we have that for any $x \in V$:
$$T^t(\underbrace{ag+h}_{\in W^*})(x) \overset{def}{=} \left((ag+h)\circ T\right)(x) = (ag+h)(T(x)) = a\cdot\left(g(T(x))\right) + h(T(x)) \overset{def}{=} \left(a \cdot T^t(g)\right)(x) + \left(T^t(h)\right)(x),$$
hence $T^t(ag+h) = aT^t(g) + T^t(h)$, so $T^t: W^* \to V^*$ is a lin. transformation.

2) Let $\beta = \{x_1, \dots, x_n\}$ and $\gamma = \{y_1, \dots, y_m\}$ be as given, with dual bases $\beta^* = \{f_1, \dots, f_n\}$ and $\gamma^* = \{g_1, \dots, g_m\}$, resp.
Let $A = [T]_\beta^\gamma = (A_{ij})$.
To find the j^{th} column of the matrix $[T^t]_{\gamma^*}^{\beta^*}$, we begin by expressing the vector $T^t(g_j) \in V^*$ as a linear combination of the vectors of β^*. By Thm 2.24 we have:
$T^t(g_j) = g_j T = \sum_{s=1}^n (g_j T)(x_s) f_s$.
So the row i, column j entry of the matrix $[T^t]_{\gamma^*}^{\beta^*}$ is
$$(g_j T)(x_i) = g_j(T(x_i)) = g_j\left(\sum_{k=1}^m A_{ki} y_k\right) = \sum_{k=1}^m A_{ki} \underbrace{\boxed{g_j(y_k)}}_{\substack{=1 \text{ if } j=k \\ =0 \text{ if } j \ne k}} = A_{ji}.$$
Hence $[T^t]_{\gamma^*}^{\beta^*} = A^t$.