PSEUDOFINITE MODEL THEORY (MATH 223M, UCLA, FALL 2019)

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Lecture notes in progress. All comments and corrections are welcome.
Last update: October 29, 2019

Contents

1. Preliminaries on pseudofiniteness

1.1. **Ultralimits.** Let (X, d) be a metric space, and let $\mathcal{U}$ be a *non-principal* ultrafilter on $\mathbb{N}$.

Definition 1.1. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points in X. The point $x \in X$ is called the *ultralimit of* x_n (relatively to $\mathcal{U}$), denoted $x = \lim_{\mathcal{U}} x_n$, if for every $\varepsilon > 0$ we have $\{n \in \mathbb{N} : d(x_n, x) \leq \varepsilon\} \in \mathcal{U}$.

Remark 1.2.
1. If an ultralimit of a sequence of points exists, then it is unique.
2. If $x = \lim_{n\to\infty} x_n$ in the usual sense of metric limits, then $x = \lim_{\mathcal{U}} x_n$ (uses that $\mathcal{U}$ is non-principal).

Fact 1.3. *If (X, d) is compact and $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$, then* ***any*** *sequence of points in X has an ultralimit relatively to $\mathcal{U}$.*

Proof. Let $(x_i : i \in \mathbb{N})$ be a sequence in X without an ultralimit, then for each $x \in X$ there is some open set $U_x \ni x$ such that $\{i \in \mathbb{N} : x_i \in U_x\} \notin \mathcal{U}$. By compactness, there is a finite sub-covering $U_1, \ldots, U_n$ of X. But the $\{i : x_i \notin U_j\} \in \mathcal{U}$ for each $1 \leq j \leq n$, hence $\emptyset = \bigcap_{j=1}^{n} \{i : x_i \notin U_j\} \in \mathcal{U}$, a contradiction. □

Corollary 1.4. *Any* bounded *sequence $(x_n : n \in \mathbb{N})$ of real numbers has a well-defined ultralimit in $\mathbb{R}$ relatively to any non-principal ultrafilter on $\mathcal{U}$ (as closed intervals are compact).*

Of course, this limit depends on the ultrafilter. For example, let $x_n = 0$ if n is even and $x_n = 1$ if n is odd. Then $\lim_{\mathcal{U}} x_n = 0$ for any ultrafilter $\mathcal{U}$ on $\mathbb{N}$ containing the set of even numbers, and $\lim_{\mathcal{U}} x_n = 1$ for any ultrafilter on $\mathbb{N}$ containing the set of odd numbers.

Exercise 1.5. Let (X, d) and (Y, d') be metric spaces, and assume that $f : X \to Y$ is continuous. Then for any sequence $(a_i)_{i \in \mathbb{N}}$ from X and any non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we have

$$\lim_{\mathcal{U}} a_i = a \implies \lim_{\mathcal{U}} f(a_i) = f(a).$$

1.2. **Ultraproducts.** For details, see e.g. [3].

Let $\mathcal{L}$ be a language and I an infinite set. Suppose that $\mathcal{M}_i$ is an $\mathcal{L}$-structure for each $i \in I$. Let $\mathcal{U}$ be an ultrafilter on I. Then we have:

Fact 1.6. *(Łoś theorem) Let $\varphi(x_1, \ldots, x_n)$ be an $\mathcal{L}$-formula, and let $\mathcal{M} = \prod_{i \in I} \mathcal{M}_i / \mathcal{U}$ be the ultraproduct of the $\mathcal{M}_i$ modulo $\mathcal{U}$. Then for any $[a_1], \ldots, [a_n] \in \mathcal{M}$,*

$$\mathcal{M} \models \varphi([a_1], \ldots, [a_n]) \iff \{i \in I : \mathcal{M}_i \models \varphi(a_1(i), \ldots, a_n(i))\} \in \mathcal{U},$$

where $[a]$ denotes the class of an element $a = (a(i) : i \in I) \in \prod_{i \in I} \mathcal{M}_i$ modulo the equivalence relation of equality $\mathcal{U}$-almost everywhere.

Corollary 1.7. *For each set of sentences T in $\mathcal{L}$, every ultraproduct of models of T is a model of T.*

Definition 1.8. Let κ be a cardinal. A structure $\mathcal{M}$ is *κ-saturated* if every partial type over a set of parameters of size $< \kappa$ in $\mathcal{M}$ is realized in $\mathcal{M}$.

Fact 1.9. *Let $\mathcal{L}$ be a countable language, $(\mathcal{M}_i : i \in \mathbb{N})$ a sequence of $\mathcal{L}$-structures and $\mathcal{U}$ a non-principal ultrafilter on $\mathbb{N}$. Then the ultraproduct $\mathcal{M} = \prod_{i \in \mathbb{N}} \mathcal{M}_i / \mathcal{U}$ is $\aleph_1$-saturated (i.e. every partial type over a* ***countable*** *set of parameters is realized in $\mathcal{M}$).*

Note that every infinite κ-saturated structure $\mathcal{M}$ has size at least κ (if $|\mathcal{M}| < \kappa$, then $\{x \neq a : a \in M\}$ is a partial type over a set of size $< \kappa$ which cannot be realized in $\mathcal{M}$). If follows from Proposition 1.9 that any ultraproduct relatively to a non-principal ultrafilter in $\mathbb{N}$ is either finite or of size at least $\aleph_1$. In fact, more is true.

Fact 1.10. [6]*Let I be an arbitrary set and $\{\mathcal{M}_i : i \in I\}$ all finite. The for any ultrafilter $\mathcal{U}$ on I, the ultraproduct $\prod_{i \in \mathbb{N}} \mathcal{M}_i / \mathcal{U}$ is either finite or of cardinality $\geq 2^{\aleph_0}$.*

Corollary 1.11. *Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$, and assume that $\mathcal{M}_i, i \in \mathbb{N}$ is a **countable** $\mathcal{L}$-structure. Then any ultraproduct $\mathcal{M} = \prod_{i\in\mathbb{N}} \mathcal{M}_i/\mathcal{U}$ is either finite or of size $2^{\aleph_0}$.*

Proof. Obviously $\left|\prod_{i\in\mathbb{N}} \mathcal{M}_i/\mathcal{U}\right| \leq \left|\prod_{i\in\mathbb{N}} \mathcal{M}_i\right| \leq \left|\mathbb{N}^{\mathbb{N}}\right| = 2^{\aleph_0}$, and $|\mathcal{M}| \geq 2^{\aleph_0}$ by Fact 1.10. □

1.3. **Pseudofiniteness (of structures, theories, subsets).**

Definition 1.12. An $\mathcal{L}$-structure $\mathcal{M}$ is *pseudofinite* if for every $\mathcal{L}$-sentence φ, $\mathcal{M} \models \varphi$ implies that there is a finite $\mathcal{M}_0$ such that $\mathcal{M}_0 \models \varphi$.
$\mathcal{M}$ is *strictly pseudofinite* if $\mathcal{M}$ is pseudofinite and not finite.

Proposition 1.13. *Let $\mathcal{L}$ be a language and $\mathcal{M}$ an $\mathcal{L}$-structure. Then the following are equivalent:*

(1) *$\mathcal{M}$ is pseudofinite.*
(2) *$\mathcal{M}$ is elementarily equivalent to an ultraproduct of finite $\mathcal{L}$-structures.*
(3) *$\mathcal{M} \models \mathrm{Fin}_{\mathcal{L}}$, where $\mathrm{Fin}_{\mathcal{L}} := \{\varphi \in \mathcal{L} : \exists \mathcal{M}$ a finite $\mathcal{L}$-structure with $\mathcal{M} \models \varphi\}$.*

Proof. (1) $\Rightarrow$ (2). Suppose $\mathcal{M}$ is pseudofinite. Let I be the collection of all finite subsets of $\mathrm{Th}(\mathcal{M})$. Given $i = \{\varphi_1, \ldots, \varphi_n\} \in I$, there exists some finite $\mathcal{M}_i \models \bigwedge_{1\leq i\leq n} \varphi_i$. For each $i \in I$, let $A_i := \{j \in I : \mathcal{M}_j \models \varphi$ for all $\varphi \in i\}$, and let $\mathcal{U}_0 := \{A_i : i \in I\}$. Then $\mathcal{U}_0$ has the finite intersection property:

$$\begin{aligned} A_i \cap A_{i'} &= \{j \in I : \mathcal{M}_j \models \varphi \text{ for all } \varphi \in i\} \cap \{j \in I : \mathcal{M}_j \models \varphi \text{ for all } \varphi \in i'\} \\ &= \{j \in I : \mathcal{M}_j \models \varphi \text{ for all } \varphi \in i \cup i'\} = A_{i\cup i'} \neq \emptyset, \end{aligned}$$

hence $\mathcal{U}_0$ can be extended to an ultrafilter $\mathcal{U}$ on I. Then $\mathcal{M} \equiv \prod_{i\in I} M_i/\mathcal{U}$. Indeed, if $\mathcal{M} \models \varphi$, then $\{i \in I : \mathcal{M}_i \models \varphi\} \supseteq A_{\{\varphi\}} \in \mathcal{U}$, hence by Łoś' theorem $\prod_{i\in\mathcal{U}} \mathcal{M}_i/\mathcal{U} \models \varphi$.

(2) $\Rightarrow$ (3). Assume $\mathcal{M} \equiv \prod_{i\in I} \mathcal{M}_i/\mathcal{U}$, where $\{\mathcal{M}_i : i \in I\}$ are all finite and $\mathcal{U}$ is an ultrafilter on I. Then for every $\varphi \in \mathrm{Fin}_{\mathcal{L}}$ and $i \in I$ we have $\mathcal{M}_i \models \varphi$, hence $\prod_{i\in I} \mathcal{M}_i/\mathcal{U} \models \varphi$, so $\mathcal{M} \models \varphi$. Hence $\mathcal{M} \models \mathrm{Fin}_{\mathcal{L}}$.

(3) $\Rightarrow$ (1). Let $\varphi \in \mathcal{L}$ be such that $\mathcal{M} \models \varphi$. If φ has no finite models, then $\mathcal{M}_0 \models \neg\varphi$ for every finite $\mathcal{L}$-structure $\mathcal{M}_0$, so $\neg\varphi \in \mathrm{Fin}_{\mathcal{L}}$, hence $\mathcal{M} \models \neg\varphi$ — a contradiction. □

Remark 1.14. In particular, pseudofiniteness is a property of $\mathrm{Th}(\mathcal{M})$.

Note that not every pseudofinite structure is *isomorphic* to an ultraproduct of finite structures. For example, if $\mathcal{L}$ is countable and $\mathcal{M}$ is a pseudofinite $\mathcal{L}$-structure (see Section 1.4 many examples), we can find a countable $\mathcal{M}' \preceq \mathcal{M}$, hence $\mathcal{M}'$ is countable and pseudofinite. But any infinite ultraproduct of finite sets has cardinality at least continuum by Fact 1.10.
However, recall:

Fact 1.15. *[Keisler-Shelah] $\mathcal{M} \equiv \mathcal{N} \iff$ there exists an ultrafilter $\mathcal{U}$ on some set I such that $\mathcal{M}^{\mathcal{U}} = \prod_{i\in I} \mathcal{M}/\mathcal{U} \cong \prod_{i\in I} \mathcal{N}/\mathcal{U} = \mathcal{N}^{\mathcal{U}}$.*

Hence

Corollary 1.16. *A structure $\mathcal{M}$ is pseudofinite if and only if it has an ultrapower isomorphic to an ultraproduct of finite structures.*

This follows from Proposition 1.13, Fact 1.15 and the following exercise.

Exercise 1.17. (1) Let $\mathcal{U}$ be an ultrafilter on I, and for every $i \in I$ let $\mathcal{V}_i$ be an ultrafilter on J_i, and let $K := \{(i,j) : i \in I, j \in J_i\}$. Let $\mathcal{W}$ be the collection of subsets of K of the form

$$A \in \mathcal{W} \iff \{i \in I : \{j \in J_i : (i,j) \in A\} \in \mathcal{V}_i\} \in \mathcal{U}.$$

Then $\mathcal{W}$ is an ultrafilter on K.

(2) Every ultraproduct of ultraproducts of finite structures is isomorphic to an ultraproduct of finite structures.

Exercise 1.18. Let $\mathcal{M}$ be a pseudofinite $\mathcal{L}$-structure.

(1) Let $\mathcal{M}$ be pseudofinite, and assume that $\mathcal{N}$ is interpretable in $\mathcal{M}$ (using formulas with parameters). Then $\mathcal{N}$ is also pseudofinite.
(2) In particular: any reduct of $\mathcal{M}$ to a sublanguage of $\mathcal{L}$, any expansion of $\mathcal{M}$ by constants and by definable relations, and $\mathcal{M}^{\mathrm{eq}}$, are pseudofinite.

Definition 1.19. Let T be a consistent (but possibly incomplete) theory in a language $\mathcal{L}$.

(1) T is *weakly pseudofinite* if whenever $T \models \varphi$, then φ is true in some finite $\mathcal{L}$-structure (not necessarily a model of T).
(2) T is *strongly pseudofinite* if whenever $T \cup \{\varphi\}$ is consistent, then there exist some finite $\mathcal{M}_0 \models \varphi$.

Example 1.20. The empty $\mathcal{L}$-theory is weakly pseudofinite, but not strongly pseudofinite.

Exercise 1.21. Let T be an $\mathcal{L}$-theory.

(1) T is weakly pseudofinite if and only if T has some pseudofinite model.
(2) T is strongly pseudofinite if and only if $T \models \mathrm{Fin}_{\mathcal{L}}$, if and only if every model of T is pseudofinite.
(3) If T is complete, then T is weakly pseudofinite if and only if T is strongly pseudofinite.

Definition 1.22. A complete theory T is *pseudofinite* if it is weakly (equivalently, strongly) pseudofinite.

Definition 1.23. Let $\mathcal{M}$ be an $\mathcal{L}$-structure, and A a subset of $\mathcal{M}^n$. We say that *a set A is pseudofinite* in $\mathcal{M}$ if whenever φ is a sentence in the language $\mathcal{L}_P = \mathcal{L} \cup \{P\}$ with an additional predicate symbol P naming A and $(\mathcal{M}, A) \models \varphi$, then there is an $\mathcal{L}$-structure $\mathcal{M}'$ and a *finite* subset A' of $(\mathcal{M}')^n$ such that $(\mathcal{M}', A') \models \varphi$.

Remark 1.24. (1) Pseudofiniteness of A in $\mathcal{M}$ is a property of $\mathrm{Th}_{\mathcal{L}_P}(\mathcal{M}, A)$.

(2) Suppose that $A \subseteq M^n$ is definable by some $\mathcal{L}$-formula $\varphi(x, b)$. The "A is pseudofinite in $\mathcal{M}$" is equivalent to: for every $\mathcal{L}$-formula $\psi(y) \in \mathrm{tp}_{\mathcal{M}}(b)$, there is an $\mathcal{L}$-structure $\mathcal{M}'$ and $b' \in \mathcal{M}'$ such that $\mathcal{M} \models \psi(b')$ and $\varphi(M, b')$ is finite.
(3) So if A is definable by an $\mathcal{L}$-formula $\varphi(x)$ without parameters, then A is pseudofinite in $\mathcal{M}$ if and only if for every $\psi \in \mathrm{Th}_{\mathcal{L}}(\mathcal{M})$ there exists some $\mathcal{L}$-structure $\mathcal{M}'$ such that $\mathcal{M} \models \psi$ and $\varphi(M')$ is finite.

Similarly to Proposition 1.13 we have:

Proposition 1.25. *Let $\mathcal{M}$ be an $\mathcal{L}$-structure and A a subset of M^n. The following are equivalent.*

(1) *A is pseudofinite in $\mathcal{M}$.*
(2) *$(\mathcal{M}, A)$ is elementarily equivalent to some ultraproduct of $\mathcal{L}_P$-structures of the form $(\mathcal{M}', A')$ with A' finite.*
(3) *$(\mathcal{M}, A) \models \mathrm{Fin}_{\mathcal{L},P}$, where $\mathrm{Fin}_{\mathcal{L},P}$ is the set of all $\mathcal{L}_P$-sentences which are true in every $\mathcal{L}_P$-structure (M', A') with A' finite.*

Exercise 1.26. Assume that $A, B \subseteq M^n$ are pseudofinite in $\mathcal{M}$. Is $A \cup B$ pseudofinite in $\mathcal{M}$?

Exercise 1.27. Let $A \subseteq M^n$ be pseudofinite in $\mathcal{M}$ and let f be a definable function (possibly with parameters in $\mathcal{M}$) such that $f(A) \subseteq A$. Then f is injective if and only if it is surjective.

1.4. **Some examples of pseudofinite theories.**

(1) For any language $\mathcal{L}$, the common theory of all $\mathcal{L}$-structures is weakly pseudofinite.
(2) Let $\mathcal{L}$ be a language and T an $\mathcal{L}$-theory. Let T_{Fin} be the common theory of all finite models of T. Then T_{Fin} is strongly pseudofinite. (Exercise: T_{Fin} may not be the weakest strongly pseudofinite theory extending T).
(3) The theory DLO is not pseudofinite (the sentence given by the conjunction of the axioms of linear order and
$$\forall x \forall y \, (x < y \to \exists z \, (x < z < y))$$
has no finite models).
Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Let $\mathcal{M}_i = (\{0, 1, \ldots, i-1\}, <)$ be a finite linear order on i elements. Let $\mathcal{M} := \prod_{i \in \mathbb{N}} \mathcal{M}_i / \mathcal{U}$, and let $T := \mathrm{Th}(\mathcal{M})$. For any $i \in \mathbb{N}$, $\mathcal{M}_i$ has the first and the last elements, and is a discrete linear order (i.e. every element has immediate successor and predecessor) of size $\geq i$. Each of these properties can be expressed by a first-order sentence. Hence, by Łoś theorem, $\mathcal{M}$ is an *infinite* discrete linear order with endpoints, and these properties axiomatize a complete first-order theory (which is then the unique complete pseudofinite theory of linear orders). In fact, $\mathcal{M} \cong \mathbb{N} + \sum_{j \in L} \mathbb{Z} + \mathbb{N}^*$, where L is a dense (by $\aleph_1$-saturation) linear order without endpoints of cardinality continuum and $\mathbb{N}^*$.
(4) The abelian group $(\mathbb{Z}, +)$ is not pseudofinite.
By Exercise 1.27, as the definable map $x \mapsto x + x$ is injective, but not surjective.
(5) Let ACF be the theory of algebraically closed fields (in the language of rings). It is not weakly pseudofinite.
Let $K \models \mathrm{ACF}$. Assume $\mathrm{char}(K) \neq 2$, then the definable map $x \mapsto x^2$ is surjective by algebraic closedeness, but not injective (as $1 \neq -1 \mapsto 1$). If $\mathrm{char}(K) \neq 3$, then the definable map $x \mapsto x^3$ has the same property. Let φ be the sentence expressing "K is a field, and either the square map or the cube map is surjective, but not injective". Then $\mathrm{ACF} \models \varphi$, but φ has no finite model.
(6) However, we have:

Proposition 1.28. *Any field has an infinite pseudofinite field extension.*

Proof. The algebraic closure of a finite field is the directed union of its finite subfields, hence it embeds into an ultraproduct of its finite subfields (namely, $\mathbb{F}_p^{\mathrm{alg}} = \bigcup_{n\in\omega} \mathbb{F}_{p^n}$ embeds into $\prod_{n\in\omega} \mathbb{F}_{p^n}/\mathcal{U}$ for any $\mathcal{U}$ an ultrafilter on ω containing the filter $\mathcal{F} = \{S_m : m \in \omega\}$, where $S_m = \{km : k \in \omega\}$, via the map $f(a) = (b_n)_{n\in\omega} \in \prod_{n\in\omega} \mathbb{F}_{p^n}$ for all $a \in \mathbb{F}_p^{\mathrm{alg}}$ with $b_n := a$ if $a \in \mathbb{F}_{p^n}^{\mathrm{alg}}$ and $b_n := 0$ otherwise), and such an ultraproduct is a pseudofinite field. Hence the claim holds for algebraically closed fields of positive characteristic, so for all algebraically closed fields (since $\prod_{p \text{ prime}} \mathbb{F}_p/\mathcal{U} \models \mathrm{ACF}_0$ for any $\mathcal{U}$ a non-principal ultrafilter on the set of prime numbers), and thus for all fields. $\square$

(7) Any vector space over any field K is pseudofinite (viewed as a structure in the language $\mathcal{L}_K = \left\{+, 0, (\cdot_r(x))_{r\in K}\right\}$ of vector spaces).
Take an infinite pseudofinite field extending K. Viewing it as a vector space over K (which is a reduct), it remains pseudofinite by Exercise 1.18. Since the $\mathcal{L}_K$-theory of infinite vector spaces over K is complete (e.g., by κ-categoricity for any $\kappa > |K|$), every vector space over K is pseudofinite.

(8) What about vector spaces over division rings?

Classical Wedderburn's theorem: every finite division ring is a field. Hence every pseudofinite division ring is a field. However if a K-vector space V is pseudofinite, it doesn't imply that K is a pseudofinite ring (since K is not interpretable in V in general, but rather is coded in the language $\mathcal{L}_K$).

Fact 1.29. [23]

(a) *Suppose K is a division ring that is finite dimensional as a vector space over its center. Then any vector space over K is pseudofinite.*
(b) *There exist division ring K and a K-vector space which is not pseudofinite.*

(9) Rado's random graph is pseudofinite.

Recall that this is a theory in the language $\mathcal{L} = \{E(x, y)\}$ with a single binary relation axiomatized by:

(a) $\forall x \neg E(x, y)$,
(b) $\forall x \forall y E(x, y) \to E(y, x)$,
(c) For each n, the nth extension axiom

$$\varphi_n := \forall x_1 \ldots \forall x_{2n} \left(\bigwedge_{1\le i\ne j\le 2n} x_i \ne y_j \right) \to \exists z \left(\bigwedge_{1\le i\le n} E(z, x_i) \wedge \bigwedge_{n<i\le 2n} \neg E(z, x_i) \right).$$

By an easy back-and-forth, this is a complete theory.
A simple probabilistic argument shows: for each n, $\mathbb{P}(G_m \models \varphi_n) \to 1$ as $m \to \infty$, where G_m is a graph on m vertices picked uniformly at random (see e.g. [19] for the details)

(10) There is a unique complete pseudofinite theory of Boolean algebras (in the language $L = (\cup, \cap, \neg, 0, 1)$). It is axiomatized by a single sentence saying that the Boolean algebra is atomic (i.e. that there is an atom under any element), plus the infinity axiom schema [20].

(11) The following is a deep result in geometric model theory, we will touch on some of the ingredients in this course.

Fact 1.30. [24, 2]*Let T be a totally categorical theory (i.e. κ-categorical for all $\kappa \ge \aleph_0$) in a countable language. Then T is pseudofinite.*

In fact, a weaker assumption "$\aleph_0$-stable, $\aleph_0$-categorical" is sufficient. Note also that this fact implies that such a T is not finitely axiomatizable. They proved additionally that such T is *quasi-finitely axiomatizable*, i.e. it is axiomatized by a single sentence plus an axiom schema expressing that the structure is infinite.)

Exercise 1.31. (1) Show that every groups embeds into an ultraproduct of its finitely generated subgroups.
Hence every locally finite group embeds into a pseudofinite group.
(2) Show that the groups of all permutations of a countably infinite set does not embed into any pseudofinite group.

1.5. **Faux finite theories.** We also consider the following strengthening of an infinite structure being "finite-like".

Definition 1.32. (1) Let $\mathcal{L}$ be a language and let $\mathcal{F}$ be an infinite family of *finite* $\mathcal{L}$-structures closed under isomorphism. We let $\lim \mathcal{F}$ be the set of all $\mathcal{L}$-sentences which are satisfied by all but finitely many structures in $\mathcal{F}$. We say that $\mathcal{F}$ is *convergent* if $\lim \mathcal{F}$ is a complete first-order theory. Equivalently, if the theory of a non-principal ultraproduct of structures in $\mathcal{F}$ does not depend on the choice of an ultrafilter.
(2) We say that a complete first-order theory T in a language $\mathcal{L}$ is *faux finite* if there is an $\mathcal{L}$-sentence φ such that the family $\mathcal{F}_\varphi$ of *all* finite models of φ is convergent and $\lim \mathcal{F}_\varphi = T$. We say that an $\mathcal{L}$-structure $\mathcal{M}$ is faux finite if $\mathrm{Th}(\mathcal{M})$ is faux finite.

Remark 1.33. (1) Every faux finite theory is obviously pseudofinite.
(2) Every pseudofinite quasi-finitely axiomatizable theory is faux finite (if T is axiomatized by a sentence φ and the axiom of infinity, then the family $\mathcal{F}_\varphi$ of all finite models of φ is convergent).

Exercise 1.34. (1) Which of the pseudofinite theories discussed above are faux finite?
(2) Give an example of a faux-finite theory which is not quasi-finitely axiomatizable.

1.6. **References.** Calculating the saturation or cardinality of an ultraproduct with an arbitrary index set is a difficult problem, even when it is an ultraproduct of finite structures — see for example [22, 13, 14]. Various observations about weak/strong pseudofiniteness for incomplete theories are mostly from [21].

It is a well-known open problem of Cherlin whether the theory of the triangle-free generic countable graph is pseudofinite, see e.g. [4, 16].

Any countably categorial theory with disjoint n-amalgamation for all $n \geq 2$ is pseudofinite, and certain theories of parametrized equivalence relations are pseudofinite [15].

For a survey on pseudofinite groups see [18]. Exercise 1.31 is from Theorem 6.4 there.

2. Pseudofinite dimensions

2.1. **Nonstandard cardinality of pseudofinite sets.** We introduce an auxiliary construction which provides one possible way to rigorously manipulate the non-standard cardinalities of pseudofinite sets. We fix a countable language $\mathcal{L}$, a collection of $\mathcal{L}$-structures $(\mathcal{M}_i : i \in I)$ and let $\mathcal{U}$ be a non-principal ultrafilter on I.

We enrich the language $\mathcal{L}$ to a language $\mathcal{L}^+$ which contains:

- all of the sorts from $\mathcal{L}$ with the corresponding functions/relations/constants on them;
- one additional sort R with the language of ordered rings and a unary function symbol log on it, plus an additional constant symbol ∞;
- for each $\mathcal{L}$-formula $\varphi(x, y)$, x, y tuples of variables, a function symbol f_φ from the sort corresponding to y to the new sort R.

Now every $\mathcal{L}$-structure $\mathcal{M}$ can be canonically expanded to an $\mathcal{L}^+$-structure $\mathcal{M}^+$ as follows:

- All the $\mathcal{L}$ sorts are interpreted in the same way as in $\mathcal{M}$;
- We interpret the sort R as $(\mathbb{R}, +, \cdot, 0, 1, <, \log)$;
- for every $b \in M_y$ we define $f_\varphi(b) := |\varphi(M, b)|$ if the set $\varphi(M, b)$ is finite, and $f_\varphi := \infty$ otherwise.

Finally, we let $\bar{\mathcal{M}} := \prod_{i \in I} \mathcal{M}_i^+ / \mathcal{U}$. This is a structure with sorts corresponding to $\mathcal{M} := \prod_{i \in I} \mathcal{M}_i / \mathcal{U}$ and $\mathbb{R}^* := \mathbb{R}^{\mathcal{U}}$, the sort for the *non-standard reals,* and each $\mathcal{L}$-definable subset $X = \varphi(M, b)$ in $\mathcal{M}$ is equipped with its *non-standard cardinality* $|X| := f_\varphi(b)$ (which is equal to the size of X if X is finite). Note that $\mathbb{R}$ embeds into $\mathbb{R}^*$ diagonally and $\mathbb{R}^* \succeq \mathbb{R}$. In particular, we are allowed to take sums, products and quotients of non-standard cardinalities, as well as compare them to each other, or to rational numbers, and all this operations and the non-standard log satisfy the usual properties by Łoś theorem.

2.2. **Convex subgroups of the non-standard reals.**

Definition 2.1. A non-empty set $S \subseteq \mathbb{R}^*$ is *convex* if whenever $s_1, s_2 \in S$ and $s_1 < r < s_2$, then also $r \in S$.

For any $a < b \in \mathbb{R}^* \cup \{-\infty, +\infty\}$, the interval $(a, b) := \{x \in \mathbb{R}^* : a < x < b\}$ is convex. For any $r \in \mathbb{R}^*$, the *monad of* r is defined as

$$\left\{ x \in \mathbb{R}^* : r - \frac{1}{n} < x < r + \frac{1}{n} \text{ for all } n \in \mathbb{N} \right\}$$

is convex (but not an interval).

Example 2.2. The following are convex subgroups of $(\mathbb{R}^*, +)$.

(1) The trivial subgroup $C = \{0\}$.
(2) The *group of infinitesimals* C_0, namely the monad of 0 in $\mathbb{R}^*$. (Note: this is the only monad which is also a subgroup of $(\mathbb{R}^*, +)$.
(3) Note that the family of all convex subgroups is closed under intersection. Hence for any non-empty subset A of $\mathbb{R}^*$, we let $C(A)$ be the smallest convex subgroup of $\mathbb{R}^*$ containing A.

Remark 2.3. Note that if C is a convex proper subgroup of $\mathbb{R}^*$, then the quotient $\mathbb{R}^*/C$ is an abelian ordered group with the order given by $x + C < y + C$ if and only if $x < y$ in $\mathbb{R}^*$.

Proposition 2.4. *Let* $0 < \alpha \in \mathbb{R}^*$ *be arbitrary.*

(1) *There exists the smallest subgroup* C_α *of* $(\mathbb{R}^*, +)$ *containing* α.

(2) *There exists a convex subgroup $C_{<\alpha}$ of $(\mathbb{R}^*, +)$ which is the largest convex subgroup of $\mathbb{R}^*$ not containing α.*
(3) *There exists a unique isomorphism of ordered abelian groups (in fact, even ordered $\mathbb{R}$-vector spaces) $\varphi : C_\alpha / C_{<\alpha} \to \mathbb{R}$ such that $\varphi(\alpha + C_{<\alpha}) = 1$.*

Proof. (1) It exists by the previous remark. More explicitly, let

$$C_\alpha := \{x \in \mathbb{R}^* : |x| < n\alpha \text{ for some } n \in \mathbb{N}\}.$$

Then $\alpha \in C_\alpha$ as $0 < \alpha < 2\alpha$ and C_α is convex (assume $s_1 < r < s_2$ and $s_1, s_2 \in C_\alpha$, let n be such that $|s_1|, |s_2| < n\alpha$, then $|r| < n\alpha$, hence $r \in C_\alpha$).

Suppose $\alpha \in C$ and C is a convex subgroup. Then $n\alpha \in C$, hence $C_\alpha \subseteq C$.

(2) Define $C_{<\alpha} := \{x \in \mathbb{R}^* : n \cdot |x| < \alpha \text{ for all } n \in \mathbb{N}\}$.

Then $\alpha \notin C_{<\alpha}$ since $1 \cdot \alpha \not< \alpha$.

If $s_1 < r < s_2$ and $s_1, s_2 \in C_\alpha$, we have for every $n \in \mathbb{N}$ that $ns_1 < nr < ns_2$, hence $n|r| < n \cdot \max\{|s_1|, |s_2|\} < \alpha$, so $r \in C_\alpha$, hence C_α is convex.

Suppose $C_{<\alpha} \subsetneq C$, where C is a convex subgroup of $\mathbb{R}^*$. If $x \in C \setminus C_\alpha$, then also $|x| \in C$, and by definition $n|x| \geq \alpha$ for some n. But then $0 < \alpha < (n+1)|x|$, hence $\alpha \in C$ as C is convex. Thus $C_{<\alpha}$ is the largest convex subgroup not containing α.

(3) Let $\phi : C_\alpha \to \mathbb{R}$ be the map defined by $\phi(\beta) = \sup\left\{q \in \mathbb{Q} : q \leq \frac{\beta}{\alpha}\right\}$. Note that $\frac{\beta}{\alpha} < n$ for some $n \in \mathbb{N}$ by (1), hence $\phi(\beta) \in \mathbb{R}$ for all $\beta \in C_\alpha$.

- ϕ is a homomorphism of ordered abelian groups: clear from the definition.
- ϕ is surjective. If $r \in \mathbb{R}$ then $n \leq r < n+1$ for some $n \in \mathbb{Z}$, hence $n\alpha \leq r\alpha < (n+1)\alpha$, hence $r\alpha \in C_\alpha$ by (1). Then $\phi(r\alpha) = \sup\left\{q \in \mathbb{Q} : q \leq \frac{r\alpha}{\alpha} = r\right\} = r$.
- $\ker \phi = \{x \in C_\alpha : \phi(x) = 0\} = \left\{x \in C_\alpha : -\frac{1}{n} < \frac{x}{\alpha} < \frac{1}{n} \text{ for all } n \in \mathbb{N}\right\} = \{x \in C_\alpha : n|x| < \alpha \text{ for all } n\} = C_{<\alpha}$ by (2).
- Then, by the groups isomorphism theorem, there is an isomorphism $\varphi : C_\alpha / C_{<\alpha} \to \mathbb{R}$ with $\varphi(\alpha + C_{<\alpha}) = \phi(\alpha) = 1$.

□

Note that all convex subgroups of $(\mathbb{R}^*, +, <)$ are linearly ordered by inclusion, and intuitively correspond to different orders of magnitude: if $C_1 \subsetneq C_2$ and $\alpha \in C_1, \beta \in C_2 \setminus C_1$, then α is infinitesimally small compared to β.

Definition 2.5. Consider $C_1 = \{r \in \mathbb{R}^* : |r| \leq n \text{ for some } n \in \mathbb{N}\}$. Note that C_1 is a convex subring of $\mathbb{R}^*$. Let $C_{<1} = \left\{r \in \mathbb{R}^* : |r| \leq \frac{1}{n} \text{ for all } n \in \mathbb{N}\right\}$, this is the maximal ideal of the ring C_1. Then the map $\phi : C_1 \to \mathbb{R}$ defined in 2.4(3) is called *the standard part map* and is denoted as $\mathrm{st} : C_1 \to \mathbb{R}$. It sends each $r \in C_1$ to the nearest real number $s \in \mathbb{R}$, that is $s \in \mathbb{R}$ and for all rational $q_1, q_2 \in \mathbb{Q}$ if $q_1 \leq r \leq q_2$, then $q_1 \leq s \leq q_2$. By 2.4(3) this is the unique ring morphism that respects the ordering, its kernel is the infinitesimals $C_{<1}$ and $C_1/C_{<1} \cong \mathbb{R}$ are isomorphic rings.

Exercise 2.6. Let $\mathcal{U}$ be an ultrafilter on I, and let $(r_i)_{i \in I}$ be a sequence of real numbers with $|r_i|$ bounded. Note that by boundness $r := (r_i)_{i \in I} / \mathcal{U} \in \mathbb{R}^*$ is a bounded non-standard real, i.e. $r \in C_1$. Then $\lim_{\mathcal{U}} r_i = \mathrm{st}(r)$.

2.3. **Pseudofinite dimensions parametrized by convex subgroups.**

Definition 2.7. Let $\bar{\mathcal{M}} := \prod_{i \in I} \mathcal{M}_i^+ / \mathcal{U}$ be as in Section 2.1, and let C be a convex subgroup of $\mathbb{R}^*$ containing $\mathbb{Z}$. Then for any non-empty $\mathcal{L}$-definable $\mathcal{U}$-pseudofinite subset A of M^n (or of some sort of M, in the multisorted setting) we define the *pseudofinite dimension of A with respect to C* as

$$\delta_C(A) := \log|A| + C,$$

i.e. the image of $\log|A|$ under the canonical projection of $\mathbb{R}^*$ onto $\mathbb{R}^*/C$. We define $\delta_C(\emptyset) := -\infty$. Hence δ_C takes values in $\mathbb{R}^*/C \cup \{-\infty, \infty\}$.

Remark 2.8. If $\varphi(x) \in \mathcal{L}(\mathcal{M})$ is a formula such that the definable set $\varphi(M)$ is pseudofinite, then we write $\delta_C(\varphi(x))$ to denote $\delta_C(\varphi(M))$.

1. The hypothesis that C contains $\mathbb{Z}$ ensures that finite sets have dimension 0, see Proposition 2.9(2).
2. This dimension in the ordered abelian group $\mathbb{R}^*/C$, rather than the usual integer-valued dimensions. The following proposition is viewed as justifying the term "dimension".

Proposition 2.9. *Let A, B be definable $\mathcal{U}$-pseudofinite subsets of M. Then the following hold for any convex subgroup C of $\mathbb{R}^*$ containing $\mathbb{Z}$.*

1. $\delta_C(A) \geq 0$ *for all non-empty A.*
2. *If $A \subseteq B$, then $\delta_C(A) \leq \delta_C(B)$.*
3. $A \neq \emptyset$ *is finite if and only if $\delta_C(A) = 0$.*
4. $\delta_C(A \times B) = \delta_C(A) + \delta_C(B)$.
5. $\delta_C(A \cup B) = \max\{\delta_C(A), \delta_C(B)\}$.
6. *If $f : A \to B$ is a definable function, then $\delta_C(f(A)) \leq \delta_C(B)$.*
   In particular, if f is a definable bijection then $\delta_C(A) = \delta_C(B)$.
7. *(Subadditivity) Let $f : A \to B$ be a definable surjective function such that for some $\beta \in \mathbb{R}^*/C$ and for all $b \in B$, $\delta_C\left(f^{-1}(b)\right) \leq \beta$. Then $\delta_C(A) \leq \delta_C(B) + \beta$.*

Proof. (1) is immediate as log is an increasing function.

(2) Note that $\log(m) \leq m \in C$ for all $m \in \mathbb{N}$. If $A = \{a_1, \ldots, a_m\} \subseteq M$ is a finite set, then $\delta_C(A) = \log|A| + C = m + C = C$.

(3) Assume $A = \prod A_i/\mathcal{U}$ and $B = \prod B_i/\mathcal{U}$ are both pseudofinite. Then

$$\{i : \text{both } A_i \text{ and } B_i \text{ are finite}\} \in \mathcal{U}.$$

For each such i, $\log(|A_i \times B_i|) = \log(|A_i||B_i|) = \log|A_i| + \log|B_i|$, hence $\delta_C(A \times B) = \log|A \times B| + C = \log|A| + \log|B| + C = \delta_C(A) + \delta_C(B)$.

(4) Assume that $\{i : |A_i| \geq |B_i|\} \in \mathcal{U}$ (the other case is symmetric). Then $|A_i \cup B_i| \leq 2|A_i|$ for $\mathcal{U}$-almost i, hence $\log|A_i| \leq \log|A_i \cup B_i| \leq \log(2|A_i|) = \log 2 + \log|A_i|$, hence $\log|A| \leq \log|A \cup B| \leq \log 2 + \log|A|$, so $\log|A| + C \leq \log|A \cup B| + C \leq \log 2 + \log|A| + C = \log|A| + C$, so $\delta_C(A) \leq \delta_C(A \cup B) \leq \delta_C(A)$ because $\log 2 \in C$.

(5) Let $f = \prod f_i/\mathcal{U}$. By Łoś, for $\mathcal{U}$-almost i we have that f_i is a function, hence $|f_i(A_i)| \leq |A_i|$, hence $|f(A)| \leq |A|$ which implies $\delta_C(f(A)) = \log|f(A)| + C \leq \log|A| + C = \delta_C(A)$.

(6) Say $\beta = r + C$ for $r \in \mathbb{R}^*$. By Łoś, for $\mathcal{U}$-almost i we have $f_i : A_i \to B_i$ is a surjective function. Then we can choose $b_i \in B_i$ such that $\left|f_i^{-1}(b_i)\right|$ is maximal among all $b \in B_i$. Then $|A_i| = \left|\bigcup_{b \in B_i} f_i^{-1}(b)\right| = \sum_{b \in B_i} \left|f_i^{-1}(b)\right| \leq \left|f_i^{-1}(b_i)\right| \cdot |B_i|$. Let $b := (b_i)/\mathcal{U} \in B$. By hypothesis $\delta_C\left(f^{-1}(b)\right) \leq \beta$, hence there exists $c \in C$ such that $\log\left|f^{-1}(b)\right| \leq r + c$. Then $|A| \leq \left|f^{-1}(b)\right| |B|$, so $\log|A| \leq \log\left|f^{-1}(b)\right| +$

$\log|B| \leq r + \log|B| + c$, so $\delta_C(A) = \log|A| + C \leq (r+C) + (\log|B| + C) = \beta + \delta_C(B)$. ☐

So for each non-infinitesimal convex subgroup of $(\mathbb{R}^*, +)$ there is a corresponding notion of "dimension", and varying C allows to distinguish between different degrees of graininess.

2.4. **Coarse and fine pseudofinite dimensions.** Two natural choices for a convex subgroup C are the smallest convex non-trivial subgroup C_{fin} given by the convex hull of $\mathbb{Z}$; and for a given $\alpha \in \mathbb{R}^*$, the largest convex subgroup not containing it.

Definition 2.10. We let C_{fin} be the convex hull of $\mathbb{Z}$, and we define $\delta_{\text{fin}} := \delta_{C_{\text{fin}}}$.

Remark 2.11. Asumptotically, given two pseudofinite sets $Y = \prod Y_i/\mathcal{U} \subseteq X = \prod X_i/\mathcal{U}$, Y has the same dimension as X if for some $k \in \mathbb{N}$, $|X_i| \leq k|Y_i|$ for $\mathcal{U}$-almost all i.

The characteristic feature of δ_{fin} is that every possible value $\alpha \in \mathbb{R}^*/C_{\text{fin}}$ for the dimension comes with a corresponding real-valued measure μ_α defined up to a scalar multiple.

It is characterized (up to a scalar multiple) by $\mu_\alpha(X) = 0$ if and only if $\delta_{\text{fin}}(X) < \alpha$, $\mu_\alpha(X) = \infty$ if and only if $\delta_{\text{fin}}(X) > \alpha$, and given any X, Y such that $\delta_{\text{fin}}(X) = \delta_{\text{fin}}(Y) = \alpha$, we have $\mu_\alpha(X) = \text{st}\left(\frac{|X|}{|Y|}\right)\mu_\alpha(Y)$, where $\text{st} : \mathbb{R}^*_{\geq 0} \to \mathbb{R}_\infty$ is the standard part homomorphism (if we fix any definable set X, we may define measure μ_X on all definable subests by $\mu_X(Y) = \text{st}\left(\frac{|X|}{|Y|}\right)$ so that $\delta_M(Y) = \delta_M(X)$ if and only if $\mu_X(Y) \in (0, \infty)$). Hence to every set we can attach a pair $\left(\delta_{\text{fin}}(X), \mu_{\delta_{\text{fin}}(X)}\right)$.

Proposition 2.12. *For definable sets X, Y in $\mathcal{M}$ we have $\delta_{\text{fin}}(X) = \delta_{\text{fin}}(Y)$ if and only if $\frac{1}{n} \leq \frac{|X|}{|Y|} \leq n$ for some $n \in \mathbb{N}_{>0}$.*

Proof. Indeed, if $|X| \geq |Y|$, then $\delta_{\text{fin}}(X) = \delta_{\text{fin}}(Y) \iff \log|X| - \log|Y| \in C_{\text{fin}} \iff \log\left(\frac{|X|}{|Y|}\right) \in C_{\text{fin}} \iff \frac{|X|}{|Y|} \in \mathcal{C}_{\text{fin}}$. ☐

Definition 2.13. Let $\alpha \in \mathbb{R}^*$.

(1) The *coarse pseudofinite dimension on $\mathcal{M}$ normalized by α* and denoted δ_α is defined to be $\delta_{C_{<\alpha}}$, and we restrict our attention to definable sets Y with $\log|Y| \in C_\alpha$.
(2) The corresponding dimension can be viewed as real valued, identifying $C_\alpha/C_{<\alpha}$ with $\mathbb{R}$ via the unique isomorphism sending α to 1 (See Proposition 2.4(3)).
(3) When $\alpha = \log|X|$ for some pseudofinite set X, we can also write δ_X instead of δ_α, and call δ_X the coarse dimension with respect to X.

Remark 2.14. By Exercise 2.6, for any definable pseudofinite set $A \subseteq M^n$ we have $\delta_\alpha(A) = \text{st}\left(\frac{\log|A|}{\alpha}\right)$, and $\delta_X(A) = \lim_{\mathcal{U}}\left(\frac{\log|A_i|}{\log|X_i|}\right)$.

Remark 2.15. Asumptotically, given two pseudofinite sets $Y = \prod Y_i/\mathcal{U} \subseteq X = \prod X_i/\mathcal{U}$, Y has the same dimension as X if for any $\varepsilon \in \mathbb{R}_{>0}$, for $\mathcal{U}$-almost all i we have $|Y_i| \geq |X_i|^{1-\varepsilon}$. And if $|Y_i| \approx |X_i|^\beta$, then $\delta_X(Y) \approx \beta$.

Remark 2.16. Typically, it is used in the following situation: $\mathcal{M}$ is an ultraproduct of finite structures, we normalize by the non-standard cardinality of the model, i.e. consider $\delta = \delta_M$ (so we want to compare our sets to the fixed set). For fine dimension, we have various sets in the picture with different powers, and we want to be able to compare between all of the simultaneously.

Example 2.17. Let $\mathcal{M}_i := (\mathbb{R}, +, <)$, let $X_i := \left\{(p,q) \in \mathbb{N}^2 : p+q < i\right\}, Y_i := \left\{1, 2, \dots, i^i\right\}$ and $\alpha_i := i$ for all $i \geq 1$. Let $\alpha := (\alpha_i)/\mathcal{U}$, then $\delta_\alpha(X) = 2$ and $\delta_\alpha(Y) = \infty$.

2.5. **Completions of ordered abelian monoids.** Let $(G, \leq)$ be a linearly ordered set. G is (Dedekind) *complete* if every non-empty subset of G which has a lower bound has a largest lower bound. For $X \subseteq G$, let

$$L(X) := \{s \in G : s \leq x \text{ for all } x \in X\},$$
$$U(X) := \{s \in G : s \geq x \text{ for all } x \in X\}.$$

If $L(X)$ has a largest element, it is denoted by $\inf X$, and if $U(X)$ has a smallest element, it is denoted by $\sup X$.

Note that $L(X)$ is downwards closed in G, $U(X)$ is upwards closed in G, and $l \leq u$ for all $l \in L(X), u \in U(X)$.

Let $C(X) := L(U(X))$. If $X \subseteq Y \subseteq G$, then $L(X) \supseteq L(Y)$ and $U(X) \supseteq U(Y)$, hence $C(X) \subseteq C(Y)$. Moreover $C(X) \supseteq X$ and $C(C(X)) = C(X)$, thus C is a closure operation on 2^S.

For $s \in G$, $C(\{s\}) = C\left(G^{\leq s}\right) = G^{\leq s}$.

Fact 2.18. [17] *The set $G^c := \{X \subseteq G : X, U(X) \neq \emptyset \wedge C(X) = X\}$ is ordered by inclusion $\leq^c$ is called the (Dedekind-Mac Neille)* completion *of G.*

1. *The linear order $(G^c, \leq^c)$ is linear and complete; if $F \subseteq G^c$ with a lower bound, then* $\inf F = \bigcap \{X : X \in F\}$.
2. *The map $s \mapsto C(\{s\}) = G^{\leq s} : G \to G^c$ is an embedding of ordered sets; identifying G with its image, for every $s \in G^c$ there are subsets X, Y of G with* $\sup X = \inf Y = s$.

Let now $G = (G, +, 0, \leq)$ be an ordered abelian monoid (i.e. the operation $+$ is associative, commutative and $x \leq y$ implies $x + z \leq y + z$ for all $x, y, z \in G$).

Let G^c be the completion of $(G, \leq)$ given by Fact 2.18. We define a binary operation $+^c$ on G^c as follows: for $A, B \in G^c$ let

$$A +^c B := C(A + B) \in G^c,$$

where $A + B = \{a + b : a \in A, b \in B\}$.

Note that G is equipped with the *order topology* which is generated by a sub-base of open sets of the form (a, ∞) and (∞, b) for all $a, b \in G$.

Definition 2.19.

1. G is *upper semi-continuous* if for each $a, b, g \in G$ with $g > a + b$ there are open neighborhoods U, V of a, b, respectively, so that $g > u + v$ for all $u \in U$ and $v \in V$.
2. G is *lower semi-continuous* if for each $a, b, g \in G$ with $g < a + b$ there are open neighborhoods U, V of a, b, respectively, so that $g < u + v$ for all $u \in U$ and $v \in V$.
3. G is *continuous* if it is both upper and lower semi-continuous.

Exercise 2.20. Every ordered abelian group is continuous.

Exercise 2.21. G is upper (lower) semi-continuous if and only if for any set $A, B \subseteq G$ such that $\inf A$ and $\inf B$ both exist, then $\inf(A+B) = \inf A + \inf B$ (respectively, if both $\sup A$ and $\sup B$ both exist, then $\sup(A+B) = \sup A + \sup B$).

Fact 2.22. [7, Ch. XI, §77] *If $G = (G, +, 0, \leq)$ is a lower semi-continuous ordered abelian monoid, then $(G^c, +^c, \leq^c)$ is a complete lower semicontinuous ordered abelian monoid with the identity element $G^{\leq 0}$, and identifying G with an ordered subset of G^c via the map $s \mapsto C(\{s\})$ we have that G is a submonoid of G^c.*

Corollary 2.23. *If $G = (G, +, 0, \leq)$ is an upper semi-continuous ordered abelian monoid, then there exists a complete upper semi-continuous ordered abelian monoid G^u such that G embeds into G^u as an ordered submonoid via $s \mapsto S^{\geq s}$, and for every $s \in G^u$ there are subsets X, Y of G with $\sup X = \inf Y = s$.*

Proof. If G is upper semi-continuous, then G' obtained from G by reversing the ordering is a lower semi-continuous ordered abelian monoid, and the rest follows by Fact □

Remark 2.24. If G is an ordered abelian group, then it is continuous, hence has both a lower completion G^c and an upper completion G^u. They are non-isomorphic in general (see [7, Ch. XI, §77]).

Fact 2.25. [5] *If G is an ordered abelian group, then for $C \in G^c$ the following are equivalent:*

1. *C is invertible in G^c;*
2. *$U(L(-C) + C) = G^{\geq 0}$;*
3. *$L(-C + U(C)) = G^{\leq 0}$;*
4. *for all $g \in G$, if $U(C) + g \subseteq U(C)$, then $g \in G^{\geq 0}$.*

In this case $L(-C)$ is the inverse of C in G^c.

Using this, the following can be demonstrated.

Fact 2.26.
1. *Let G be a complete archimedean ordered abelian group. Then G is isomorphic to one of $\{0\}, \mathbb{Z}, \mathbb{R}$.*
2. *Let G be a complete ordered abelian group. Then any subgroup H of G is archimedean.*
3. *If G is an archimedean ordered abelian group, then so is G^c.*

2.6. **Extending pseudofinite dimensions to type-definable sets.** Assume that $\mathcal{L}$ is countable, and that $\mathcal{M}$ is given by an ultraproduct over a countable index set. Then $\mathcal{M}$ is $\aleph_1$-saturated (Proposition 1.9). By a *type-definable set*, or $\bigwedge$*-definable set*, we mean a subset of M^n given by an arbitrary conjunction of formulas over a countable set $A \subseteq M$ of parameters.

Definition 2.27. For any type-definable set $X \subseteq M^n$ and C a convex subgroup of $\mathbb{R}^*$, we define

$$\delta_C(X) := \inf \{\delta_C(D) : X \subseteq D \text{ and } D \text{ is definable}\},$$

where the infimum is evaluated in the complete ordered abelian monoid $(\mathbb{R}^*/C)^u$ (see Section 2.5).

Remark 2.28. If $X = \bigcap_{n \in \mathbb{N}} X_n$ with X_n definable and such that $X_0 \supseteq X_1 \supseteq \ldots$, then $\delta(X) = \inf \{\delta(X_n)\}$. Indeed, $\delta(X) \leq \inf \{\delta(X_n)\}$ is clear. On the other hand, by $\aleph_1$-saturation of $\mathcal{M}$ for any definable set D with $X \subseteq D$ we must have

$X_n \subseteq D$ for all n large enough, hence $\delta(X_n) \leq \delta(D)$, so $\inf\{\delta(X_n)\} \leq \delta(X)$. (The same is true if we only assume the X_n's to be type-definable).

Lemma 2.9 holds for type-definable sets as well, where to get subadditivity we need a mild definability assumption on C satisfied by both fine and coarse pseudofinite dimensions.

Definition 2.29. We say that δ_C is *weakly type-definable* if for any $\beta \in \mathbb{R}^*/C$ and formula $\varphi(x, y) \in \mathcal{L}$, at least one of the sets $\{b \in M_y : \delta_C(\varphi(M, b)) > \beta\}$ or $\{b \in M_y : \delta_C(\varphi(M, b)) \geq \beta\}$ is given by a countable intersection of definable sets in $\mathcal{M}^+$ (see Section 2.1).

Example 2.30.
1. Assume that $C = \bigcup_{k\in\mathbb{N}} C_k$, with $C_1 \subseteq C_2 \subseteq \ldots$ definable subsets of $\mathbb{R}^*$, and say $\beta = \beta' + C$ for some $\beta' \in \mathbb{R}^*$. Then $\delta_C(\varphi(x, b)) > \beta \iff \log(|\varphi(x, b)|) > \beta' + C \iff \bigwedge_{k\in\mathbb{N}} \forall c\, (c \in C_k \to \log(\varphi(x, b)) > \beta' + c) \iff \bigwedge_{k\in\mathbb{N}} \forall c\, (c \in C_k \to \log(f_\varphi(b)) > \beta' + c)$, and this last condition is an $\mathcal{L}^+$-partial type in $\mathcal{M}^+$.
2. In particular, $C_{\text{fin}} = C_1 = \bigcup_{n\in\mathbb{N}} \{x \in \mathbb{R}^* : |x| < n\}$, hence the fine dimension δ_{fin} is weakly type-definable.
3. Assume that $C = \bigcap_{k\in\mathbb{N}} C_k$, and say $\beta = \beta' + C$ for some $\beta' \in \mathbb{R}^*$. Then $\delta_C(\varphi(x, b)) \geq \beta \iff \log(|\varphi(x, b)|) + C \geq \beta' + C \iff \exists c \left(\bigwedge_{k\in\mathbb{N}} c \in C_k \wedge \log(f_\varphi) \geq \beta' + c\right)$, and this last condition is an $\mathcal{L}^+$-partial type in $\mathcal{M}^+$.
4. In particular, $C_{<\alpha} = \bigcap_{n\in\mathbb{N}} \{x \in \mathbb{R}^* : n \cdot |x| < \alpha\}$ is of this form, hence δ_α is weakly type-definable.

Proposition 2.31. *Let X, Y be type-definable subsets of some sorts in $\mathcal{M}$.*
1. *If $X \subseteq Y$, then $\delta_C(X) \leq \delta_C(Y)$.*
2. *$X \neq \emptyset$ is finite then $\delta_C(X) = 0$.*
3. *$\delta_C(X \times Y) = \delta_C(X) + \delta_C(Y)$.*
4. *$\delta_C(X \cup Y) = \max\{\delta_C(X), \delta_C(Y)\}$.*
5. *(Subadditivity) Assume that δ_C is weakly type-definable.*
   Let f be a definable function such that for some $\gamma \in (\mathbb{R}^*/C)^u$ and for all a in $f(X)$ we have $\delta_C\left(f^{-1}(a) \cap X\right) \leq \gamma$. Then $\delta_C(X) \leq \delta_C(f(X)) + \gamma$.

Proof. (3) By $\aleph_1$-saturation any definable set $Z \supseteq X \times Y$ must contain $X' \times Y'$ for some definable $X' \supseteq X, Y' \supseteq Y$. Hence, using upper semicontinuity of $(\mathbb{R}^*/C)^u$ and Proposition 2.9(4), we have

$$\begin{aligned}
&\delta_C(X \times Y) \\
=&\inf\{\delta_C(X' \times Y') : X \subseteq X', Y \subseteq Y' \text{ and } X', Y' \text{ are definable}\} \\
=&\inf\{\delta_C(X') + \delta_C(Y') : X \subseteq X', Y \subseteq Y' \text{ and } X', Y' \text{ are definable}\} \\
=&\inf\{\delta_C(X') : X \subseteq X', X' \text{ definable}\} + \inf\{\delta_C(Y') : Y \subseteq Y', Y' \text{ definable}\} \\
=&\delta_C(X) + \delta_C(Y).
\end{aligned}$$

(4) similar to (3).

(5) As X is type-definable, it can be written as $X = \bigcap_{n\in\mathbb{N}} X_n$ with X_n a descending sequence of definable sets. Then $f(X) = \bigcap_{n\in\mathbb{N}} f(X_n)$ by $\aleph_1$-saturation of $\mathcal{M}$.

Claim. For every $\beta \in \mathbb{R}^*/C$ with $\beta > \gamma$ there is some $n(\beta) \in \mathbb{N}$ such that: $\delta\left(f^{-1}(a) \cap X_{n(\beta)}\right) \leq \beta$ for all $a \in f\left(X_{n(\beta)}\right)$.

Proof. Say $\beta = \beta' + C$ for some $\beta' \in \mathbb{R}^*$. Assume that the claim doesn't hold, then for each n there exists some $a_n \in f(X_n)$ such that $\delta\left(f^{-1}(a_n) \cap X_n\right) > \beta$.

By weak type-definability of δ, for each $n \in \mathbb{N}$ we let $\pi_n(y)$ be the partial $\mathcal{L}^+$-type such that $\pi_n(a)$ holds if and only if $\delta\left(f^{-1}(a) \cap X_n\right) > \beta$ (respectively, $\delta\left(f^{-1}(a) \cap X_n\right) \geq \beta$). And let $\pi(y) := \bigcup_{n \in \mathbb{N}} \pi_n(y)$. Then the a_n's witness that π is finitely consistent, hence can be realized by some $a \in \mathcal{M}_y$ by saturation of $\mathcal{M}^+$, hence $a \in f(X)$ and $\delta\left(f^{-1}(a) \cap X_n\right) > \beta$ (respectively, $\delta\left(f^{-1}(a) \cap X_n\right) \geq \beta$) for all $n \in \mathbb{N}$ simultaneously. In either case, $\delta\left(f^{-1}(a) \cap X\right) = \inf\left\{\delta\left(f^{-1}(a) \cap X_n\right)\right\} \geq \beta > \gamma$ — contradicting the assumption on f. □

So for each $\beta \in B$ we have $\delta\left(X_{n(\beta)}\right) \leq \delta\left(f\left(X_{n(\beta)}\right)\right) + \beta$ by Proposition 2.9(7). Thus, using upper semi-continuity of $(\mathbb{R}^*/C)^u$, we have $\delta(X) = \inf\{\delta(X_n) : n \in \mathbb{N}\} \leq \inf\left\{\delta\left(X_{n(\beta)}\right) : \beta \in B\right\} \leq \inf\left\{\delta\left(f\left(X_{n(\beta)}\right)\right) + \beta : \beta \in B\right\} = \inf\left\{\delta\left(f\left(X_{n(\beta)}\right)\right) : \beta \in B\right\} + \inf\{\beta : \beta \in B\} = \delta(f(X)) + \gamma$. □

Problem 2.32. Does subadditivity hold for an arbitrary convex subgroup C without any definability assumptions on it?

Definition 2.33. If a is a finite tuple and $B \subseteq M$ is countable, we define

$$\delta_C(a/B) := \delta_C(\mathrm{tp}(a/B)) = \inf\{\delta_C(D) : a \in D, D \text{ definable over } B\}.$$

We will write $\delta_C(a)$ for $\delta_C(a/\emptyset)$.

Note that if $B_1 \subseteq B_2$, then $\delta_C(a/B_2) \leq \delta_C(a/B_1)$. If X is type definable over B then $\delta_C(a/B) \leq \delta_C(X)$ for every $a \in X$. Saturation allows to always find a tuple of full dimension:

Proposition 2.34. *(Existence of independent realizations) If $X \subseteq M^n$ is $\bigwedge$-definable over B, then X contains some element a with $\delta_C(a/B) = \delta_C(X)$.*

Proof. For any $a \in X$ by definition we have $\delta_C(a/B) < \delta_C(X)$ if and only if there is some B-definable set Z with $a \in Z$ and $\delta(Z) < \delta(X)$ (evaluated in $(\mathbb{R}^*/C)^c$). Consider the family of all B-definable subsets of Z with $\delta_C(Z) < \delta_C(X)$. It is enough to show that their union does not contain X. If it did, by $\aleph_1$-saturation X would be contained in the union of finitely many of them, say $Z_1, \ldots, Z_m$, so $\delta_C\left(\bigcup_{i=1}^m Z_i\right) \geq \delta_C(X)$ by monotonicity. But $\delta_C\left(\bigcup_{i=1}^m Z_i\right) = \max\{\delta_C(Z_i)\} < \delta_C(X)$, a contradiction. □

Proposition 2.35. *For a tuple $a = (a_1, \ldots, a_n) \in M^n$ and a countable set B, $\delta_C(a/B)$ depends only on the set of the coordinates $\{a_1, \ldots, a_n\} \subseteq M$.*

Proof. Indeed $\delta_C(a/B)$ is invariant under permutation of the coordinates of a because these induce bijections of M^n, and thus preserve δ_C-dimension by Proposition 2.9(6).

If X is a definable set in M^n such that the last two coordinates x_{n-1} and x_n coincide for all $x \in X$, then $\delta_C(X) = \delta_C(\pi(X))$ where $\pi(X)$ is the projection onto the first $n-1$ coordinates which is a bijection in this case, again by 2.9(6) (and this claim can be iterated). □

2.7. Continuity and additivity of pseudofinite dimensions.

Definition 2.36. (1) We say that δ_C is *invariant* if for every formula $\varphi(x, y)$ and every $b \in M_y$, $\delta_C(\varphi(x, b))$ depends only on $\mathrm{tp}(b)$ (this holds for example if δ_C is $\mathrm{Aut}(\mathcal{M})$-invariant).

(2) Assume δ_C invariant. We say that δ_C is *continuous* if for every formula $\varphi(x,y)$, the map

$$S_y(\emptyset) \to \mathbb{R}^*/C \cup \{-\infty,+\infty\} : \operatorname{tp}(b) \mapsto \delta_C(\varphi(x,b))$$

from the space of types over $\emptyset$ to the two-point compactification of the order topology on $\mathbb{R}^*/C$ is continuous (note that this map is well-defined by invariance).

(3) We say that δ_C is *definable* if δ_C is continuous and the set $\{\delta_C(\varphi(x,b)) : b \in M_y\}$ is finite for any $\varphi(x,y) \in \mathcal{L}$.

Exercise 2.37. Let $\delta = \delta_\alpha$ for some $\alpha \in \mathbb{R}^*$. Then the following are equivalent.

(1) δ_α is invariant and continuous in $\mathcal{M}$.
(2) For any Y an $\emptyset$-definable subset of $M^n \times M^m$ and $\beta < \gamma \in \mathbb{R}$, there is some $\emptyset$-definable set $W \subseteq M^m$ such that $\{b \in M^m : \delta(Y_b) \leq \beta\} \subseteq W \subseteq \{b \in M^m : \delta(Y_b) < \gamma\}$.
(3) For any $n, m \geq 1$, $\beta \in \mathbb{R}$, $\varepsilon \in \mathbb{R}_{>0}$ and an $\emptyset$-definable set $Y \subseteq M^n \times M^m$ there is an $\emptyset$-definable set $W \subseteq M^m$ such that

$$\{b \in M^m : \delta(Y_b) \geq \beta + \varepsilon\} \subseteq W \subseteq \{b \in M^m : \delta(Y_b) \geq \beta\},$$

where $Y_b := \{a \in M^m : (a,b) \in Y\}$ is the fiber of Y at b.

Exercise 2.38. δ_C is definable if and only if for every $\varphi(x,y) \in \mathcal{L}$ and $b \in M_y$, there is some $\theta(y) \in \operatorname{tp}(b)$ such that $\mathcal{M} \models \theta(a) \iff \delta_C(\varphi(x,b)) = \delta_C(\varphi(x,a))$ for all $a \in M_y$.

Proposition 2.39. *Let $\delta = \delta_\alpha, \alpha \in \mathbb{R}^*$ be a coarse dimension. Given any $\mathcal{M} = \prod_{i\in\mathbb{N}} \mathcal{M}_i$ in a countable language $\mathcal{L}$, there exists an expansion $\mathcal{M}' = \prod_{i\in\mathbb{N}} \mathcal{M}'_i$* in a countable language $\mathcal{L}' \supseteq \mathcal{L}$ such that δ is continuous in $\mathcal{M}'$, and $\mathcal{M}'_i$ is an expansion of $\mathcal{M}_i$.

Proof. Write $\alpha = (\alpha_i)/\mathcal{U}, \alpha_i \in \mathbb{R}$. Let $\mathcal{L}_0 := \mathcal{L}$, and let $\mathcal{L}_{i+1}$ be obtained from $\mathcal{L}_i$ by adding a new predicate $\psi_{\varphi(x,y),q}(y)$ for each formula $\varphi(x,y) \in \mathcal{L}_i$ and each $q \in \mathbb{Q}$. Let $\mathcal{L}' := \bigcup_{i\in\mathbb{N}} \mathcal{L}_i$, $|\mathcal{L}'| = |\mathcal{L}| + \aleph_0$. For each i, we define an $\mathcal{L}'$-expansion $\mathcal{M}'_i$ of $\mathcal{M}_i$ by interpreting $\psi_{\varphi(x,y),q}(M_i) := \left\{b \in (M_i)_y : |\varphi(M,b)|^q \geq \alpha_i\right\}$. Then in the ultraproduct $\mathcal{M}'$ we have $\psi_{\varphi(x,y),q} = \left\{b : |\varphi(M',b)|^q \geq \alpha\right\}$.

We claim that δ is continuous in $\mathcal{M}'$. Let $\alpha \in \mathbb{R}, \varepsilon \in \mathbb{R}_{>0}$ be arbitrary. Pick a rational $q \in (\alpha, \alpha+\varepsilon)$. Then if $b \in \psi_{\varphi(x,y),q}(M')$ then $|\varphi(M,b)|^q \geq \alpha$, so $\delta(\varphi(M,b)) \geq q > \alpha$. And if $\delta(\varphi(M,b)) \geq \alpha + \varepsilon$, then $\delta(\varphi(M,b)) \geq q$ and $b \in \psi_{\varphi(x,y),q}(M)$, hence condition (3) in Exercise 2.37 is satisfied. $\square$

Exercise 2.40. Show that for $\delta = \delta_X$ with X an $\mathcal{L}$-definable set, we can alternatively achieve continuity by adding *cardinality comparison predicates*, i.e. for each formula $\varphi(x,y)$ add a new predicate $\theta(y,y')$ interpreted in $\mathcal{M}_i$ as

$$\left\{(a,a') \in (M_i)_y \times (M_i)_y : |\varphi(M_i,a)| \leq |\varphi(M_i,a')|\right\}$$

(and closing under these in countably many steps).

Remark 2.41. The continuity of $\delta = \delta_\alpha$ automatically extends to definable sets with parameters. Namely, if in Exercise 2.37(3) Y is assumed to be A-definable for some set of parameters $A \subseteq M$ and δ is continuous, we may find a A-definable W satisfying the required condition. Indeed, given an A-definable $Y \subseteq M^n \times M^m$, there exist some l, some $\emptyset$-definable set $Y^0 \subseteq M^{n+m+l}$ and a tuple $a_0 \in A^l$ so that $Y = Y^0_{a_0} = \left\{(x,y) : (x,y,a_0) \in Y^0\right\}$. By continuity of δ, there exists some $\emptyset$-definable set $W^0 \subseteq M^m$ such that

$$\left\{(b,a) \in M^{m+l} : \delta\left(Y_{b,a}\right) \geq \beta + \varepsilon\right\} \subseteq W^0 \subseteq \left\{(b,a) \in M^{m+l} : \delta\left(Y_{b,a}\right) \geq \beta\right\}.$$

But then $W := W^0_{a_0}$ satisfies the same condition with respect to the fibers of Y.

The following result strengthens sub-additivity of δ from 2.31 and is a desirable characteristic of a dimension function (shared by dimension in vector spaces, transcendence degree, Morley rank in strongly minimal theories, etc).

Proposition 2.42. *If $\delta = \delta_\alpha$ is continuous, then it is additive: $\delta(ab/B) = \delta(b/B) + \delta(a/bB)$.*

Proof. Given any $\varepsilon \in \mathbb{R}_{>0}$, by definition of δ_α for types there exist some B-definable sets $Y, Y' \subseteq M^n \times M^m$ and $Z \subseteq M^m$ such that $\delta_\alpha(ab/B) \leq \delta_\alpha(Y) \leq \delta_\alpha(ab/B) + \varepsilon$ and $\delta_\alpha(a/bB) \leq \delta_\alpha(Y'_b) \leq \delta_\alpha(a/bB) + \varepsilon$, and $\delta_\alpha(b/B) \leq \delta_\alpha(Z) \leq \delta(b/B) + \varepsilon$. Replacing both Y, Y' by $Y \cap Y' \cap \pi_2^{-1}(Z)$, we may assume that $Y = Y'$ and $Z = \pi_2(Y)$. Now by continuity of δ_α there exists a B-definable set $W \ni b$ such that $|\delta(Y_{b'}) - \delta(Y_b)| < \varepsilon$ for all $b' \in W$. Further replacing Y by $Y \cap \pi_2^{-1}(W)$ we may assume $\delta(ab/B) \leq \delta(Y) \leq \delta(ab/B) + \varepsilon$, $\delta(b/B) \leq \delta(\pi_2(Y)) \leq \delta(b/B) + \varepsilon$ and all fibers $Y_{b'}$ with $b' \in \pi_2(Y)$ satisfy $\delta(a/bB) - \varepsilon \leq \delta(Y_{b'}) \leq \delta(a/bB) + \varepsilon$. Thus $|\delta(ab/B) - \delta(b/B) - \delta(a/bB)| \leq 3\varepsilon$, as desired. $\square$

Exercise 2.43. We remark that additivity implies subadditivity as in Proposition 2.31.

First note that if X, Y are B-definable sets and $f : X \to Y$ is a B-definable function, then for any $a \in X$ we have $\delta(a, f(a)/B) \leq \delta(a/B) + \delta(f(a)/aB) = \delta(a/B)$ (as by definition $\delta(f(a)/aB) = \inf\{\delta(\varphi(x)) : \models \varphi(f(a)) \text{ and } \varphi \text{ is } aB\text{-definable}\}$, and $\delta(f(x) = a) = 0$ by Proposition 2.9(3) as $f(x) = a$ has only one solution).

Now let $a \in X$ be such that $\delta(a/B) = \delta(X)$ by Proposition 2.34. Then, using additivity, $\delta(X) = \delta(a/B) = \delta(a, f(a)/B) \leq \delta(f(a)/B) + \delta(a/f(a)B) \leq \delta(Y) + \delta\left(f^{-1}(f(a))\right)$, as wanted.

2.8. **Independence relations arising from pseudofinite dimensions.**

Definition 2.44. We write $a \mathop{\smile\!\!\!\!\!\mid}^{\delta}_{B} b$ if $\delta(a/bB) = \delta(a/B)$.

Exercise 2.45. $a \not\mathop{\smile\!\!\!\!\!\mid}^{\delta}_{B} b \iff$ there is a formula $\theta(x) \in \operatorname{tp}(a/bB)$ such that for all $\psi(x) \in \operatorname{tp}(a/B)$ we have $\delta(\theta(x)) < \delta(\psi(x))$.

Proposition 2.46. *(Properties of coarse independence) Let $\delta = \delta_\alpha$ for some $\alpha \in \mathbb{R}^*$.*

(1) Symmetry (assuming additivity): $a \mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A} b \iff b \mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A} a$
(2) Transitivity: if $A \subseteq D \subseteq B$, then $a \mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A} B \iff \left(a \mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A} D \text{ and } a \mathop{\smile\!\!\!\!\!\mid}^{\delta}_{D} B\right)$.
(3) Extension: assume $a \mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A} b$. Then for any c there exists a' with $\operatorname{tp}(a'/Ab) = \operatorname{tp}(a/Ab)$ and $a' \mathop{\smile\!\!\!\!\!\mid}^{\delta}_{A} bc$.

Proof. Transitivity: immediate from definitions.

Symmetry: it suffices to show that $a \mathop{\smile\!\!\!\!\!\!\;|}^{\delta_\alpha}_A b \implies b \mathop{\smile\!\!\!\!\!\!\;|}^{\delta_\alpha}_A a$. Suppose $a \mathop{\smile\!\!\!\!\!\!\;|}^{\delta_\alpha}_A b$, that is $\delta_\alpha(a/bA) = \delta(a/A)$. By additivity and invariance under permutation (propositions 2.35 and 2.42) we have $\delta(b/A) + \delta(a/bA) = \delta(ab/A) = \delta(a/A) + \delta(b/aA)$, hence $\delta(b/aA) = \delta(b/A)$ as required.

Extension: same as existence (see Proposition 2.34). □

Exercise 2.47. Show that 3-amalgamation fails.

2.9. **References.** Pseudofinite dimensions were introduced in [12], and further investigated in [10, Section 5] and [11] — most of the material here is based on these papers. The extension of δ_C to type-definable sets is from [10, Section 5]. I thank Matthias Aschenbrenner for clarifiying completions of ordered abelian groups and providing the references in Section 2.5. I also used [8]. Proposition 2.4 has a nice general treatement in the context of Hahn spaces [1, Section 2].

For more on connections of forking and fine pseudofinite dimension see [9]. Coarse dimension is particularly well-behaved in sufficiently fast growing pseudofinite difference fields [25].

3. Erdős-Hajnal conjecture for stable hypergraphs

Main applications to asymptotic combinatorics stem from establishing some additional properties of δ restricting to a tame class of theories (such as strongly minimal, stable, NIP, etc.) or just tame formulas, and the results are usually in the form of comparing the dimension δ (δ-independence) with some natural notion of dimension (respectively, independence relation) in the corresponding context.

3.1. **Erdős-Hajnal conjecture.** By a graph G we mean, as usual, a pair(V, E), where E is a symmetric subset of $V \times V$ not intersecting the diagonal. If G is a graph then a *clique* in G is a set of vertices all pairwise adjacent, and an *anti-clique* in G is a set of vertices such that any two different vertices from it are non-adjacent.

As usual, for a graph H we say that a graph G is H-free if G does not contain an *induced* subgraph isomorphic to H (i.e. a substructure in the language).

It is well-known that every graph on n vertices contains either a clique or an anticlique of size $\frac{1}{2}\log n$, and that this is optimal in general. However, the following famous conjecture of Erdős and Hajnal says that one can do much better in a family of graphs omitting a certain fixed graph H.

Conjecture 3.1. *(Erdős-Hajnal conjecture) For every finite graph H there is a real number $\delta = \delta(H) > 0$ such that every finite H-free graph $G = (V, E)$ contains either a clique or an anti-clique of size at least $|V|^\delta$.*

Remark 3.2. It is known to hold for some choices of H, but is widely open in general. A variation of this conjecture starts with a finite set of finite graphs $\mathcal{H} = \{H_1, \ldots, H_k\}$ and asks for the existence of a real constant $\delta = \delta(\mathcal{H}) > 0$ such that every finite graph G which is $\mathcal{H}$-free (that is, omits all of the $H_i \in \mathcal{H}$ *simultaneously*), contains either a clique or an anti-clique of size at least $|V|^\delta$.

Definition 3.3.
(1) Given $m \in \mathbb{N}$, sets V_1, V_2 and a relation $R \subseteq V_1 \times V_2$, we say that R has the *m-order property* if there are some vertices $a_1, \ldots a_m \in V_1, b_1, \ldots, b_m \in V_2$ such that $a_i R b_j$ holds if and only if $i < j$.
(2) A graph $G = (V, E)$ has the m-order property if the relation $E \subseteq V \times V$ does.

(3) A relation is *stable* if it doens't have the m-order property for some $m \in \mathbb{N}$.

The aim of this section is to prove the following theorem.

Theorem 3.4. *For every $m \in \mathbb{N}$ there is a constant $\delta = \delta(m) \in \mathbb{R}_{>0}$ such that every finite graph $G = (V, E)$ without the m-order property contains either a clique or an anti-clique of size at least $|V|^{\delta}$.*

Theorem 3.4 implies an instance of Conjecture 3.1 for certain $\mathcal{H}$. We consider the following graphs, for each $m \in \mathbb{N}$.

(1) Let H_m be the *half-graph* on $2m$ vertices. Namely, the vertices of H_m are $\{a_1, \ldots, a_m, b_1, \ldots, b_m\}$, and the edges are $\{(a_i, b_j) : i < j\}$.
(2) Let H'_m be the complement graph of H_m. Namely, the vertices of H'_m are $\{a_1, \ldots, a_m, b_1, \ldots, b_m\}$, and the edges are $\{(a_i, b_j) : i \geq j\} \cup \{(a_i, a_j) : i \neq j\} \cup \{(b_i, b_j) : i \neq j\}$.
(3) Let H''_m have $\{a_1, \ldots, a_m, b_1, \ldots, b_m\}$ as its vertices, and $\{(a_i, b_j) : i < j\} \cup \{(a_i, a_j) : i \neq j\}$ as its edges.

Finally, let $\mathcal{H}_m = \{H_m, H'_m, H''_m\}$.

Corollary 3.5. *For every $m \in \mathbb{N}$, the Erdős-Hajnal conjecture holds for the family of all $\mathcal{H}_m$-free graphs.*

Proof. In view of Theorem 3.4, it is enough to show that for every $m \in \mathbb{N}$ there is some $m' \in \mathbb{N}$ such that if a finite graph G is $\mathcal{H}_m$-free, then it doesn't have the m'-order property. Assume that G has the m'-order property. That is, there are some vertices $a_1, \ldots, a_{m'}, b_1, \ldots, b_{m'}$ in V such that $a_i E b_j$ holds if and only if $i < j$. If m' is large enough with respect to m, by Ramsey theorem we can find some subsequences $A = \{a_{i_1}, \ldots, a_{i_{m+1}}\}$ and $B = \{b_{j_1}, \ldots, b_{j_{m+1}}\}$, $1 \leq i_1 < \ldots < i_{m+1} \leq m', 1 \leq j_1 < \ldots < j_{m+1} \leq m'$, such that each of A, B is either a clique or an anti-clique. If both are anti-cliques, then the graph induced on $(A \cup B) \setminus \{a_{i_{m+1}}, b_{j_{m+1}}\}$ is isomorphic to H_m. If both are cliques, let $a'_k := b_{j_{k+1}}$ and $b'_l := a_{i_l}$ for $1 \leq k, l \leq m$. Then the graph induced on $\{a'_1, \ldots, a'_m, b'_1, \ldots, b'_m\}$ is isomorphic to H'_m. If A is a clique and B is an anti-clique, then the graph induced on $(A \cup B) \setminus \{a_{i_{m+1}}, b_{j_{m+1}}\}$ is isomorphic to H''_m. Finally, if A is an anti-clique and B is a clique, let $a'_k := b_{j_{m+1-k}}$ and $b'_l := a_{i_{m+1-l}}$ for1 $\leq k, l \leq m$. Then the graph induced on $\{a'_1, \ldots, a'_m, b'_1, \ldots, b'_m\}$ is again isomorphic to H''_m. In any of the cases, G is not $\mathcal{H}_m$-free. ☐

3.2. **Stable relations, Shelah's 2-rank and definability of types.** We review some basic facts from local stability theory. A partitioned formula formula $\varphi(x, y)$ has the m-order property (is stable) if the relation that it defines on $M_x \times M_y$ is.

Exercise 3.6. Let $\phi(x, y), \psi(x, z)$ be stable formulas (where y, z are not necessarily disjoint tuples of variables). Then:

(1) Let $\phi^*(y, x) := \phi(x, y)$, i.e. we switch the roles of the variables. Then $\phi^*(y, x)$ is stable.
(2) $\neg\phi(x, y)$ is stable.
(3) $\theta(x, yz) := \phi(x, y) \wedge \psi(x, z)$ and $\theta'(x, yz) := \phi(x, y) \vee \psi(x, z)$ are stable.
(4) If $y = uv$ and $c \in M_v$ then $\theta(x, u) := \phi(x, uc)$ is stable.

Definition 3.7. Let $\phi(x, y)$ be a formula, by a *complete ϕ-type* over a set of parameters $A \subseteq M_y$ we mean a maximal consistent collection of formulas of the form $\phi(x,b), \neg\phi(x,b)$ where b ranges over A. Let $S_\phi(A)$ be the space of all complete ϕ-types over A.

Definition 3.8. (1) Let $\phi(x,y) \in \mathcal{L}$ be given. A type $p(x) \in S_\phi(A)$ is *definable over B* if there is some $\mathcal{L}(B)$-formula $\psi(y)$ such that for all $a \in A$,

$$\phi(x,a) \in p \iff \models \psi(a).$$

(2) A type is *definable* if it is definable over its domain.
(3) We say that types in T are *uniformly definable* if for every $\phi(x,y)$ there is some $\psi(y,z)$ such that every type can be defined by an instance of $\psi(y,z)$, i.e. if for any A and $p \in S_\phi(A)$ there is some $b \in A$ such that $\phi(x,a) \in p \iff \models \psi(a,b)$, for all $a \in A$.

Remark 3.9. Another way to think about it: Given a set $A \subseteq \mathbb{M}_x$, we say that a subset $B \subseteq A$ is *externally definable* (as a subset of A) if there is some definable (over $\mathbb{M}$) set X such that $B = X \cap A$. Assume moreover that we have $X = \phi(\mathbb{M}, c)$ above. Then $\mathrm{tp}_\phi(c/A)$ is definable if and only if B is in fact internally definable, i.e. $B = A \cap Y$ for some A-definable set Y. A set is called *stably embedded* if every externally definable subset of it is internally definable.

Example 3.10. Consider $(\mathbb{Q},<) \models \mathrm{DLO}$, and let $p = \mathrm{tp}(\pi/\mathbb{Q})$ $(\pi \in \mathbb{R} \succ \mathbb{Q})$. It is easy to check by QE that p is not definable.

Proposition 3.11. *Let $\varphi(x,y)$ be stable and $q(x) \in S_\varphi(M)$ be φ-type. Then q is defined by a positive Boolean combination of formulas of the form $\varphi^*(y,c) = \varphi(c,y)$ with c in M.*

Proof. Suppose towards contradiction that there is no finite subset of M such that q is φ^*-definable over it.

Then for all $n \in \omega$ we can construct inductively $(b_n, b'_n)_{n<\omega}$ and $(c_n)_{n<\omega}$ in M with $c_n \models q|_{c_{<n}}$ such that:
(1) $\phi(x,b_i)$ and $\neg\phi(x,b'_i)$ belong to p for every $i \in \omega$,
(2) $\phi(c_i,b_j) \to \phi(c_i,b'_j)$ holds for every $i<j$,
(3) $\phi(c_i,b_j)$ and $\neg\phi(x_i,c'_j)$ hold for every $i \geq j$.

Assume we have constructed $(b_i, b'_i, c_i : i<n)$. As q is not definable by a positive Boolean combination of the formulas $\phi^*(c_i,y)$ for $i<n$, there are some tuples b_n, b'_n in M such that $\phi(x,b_n) \in p$, $\phi(x,b'_n) \notin p$ and $\phi(c_i,b_n) \to \phi(c_i,b'_n)$ for all $i<n$. Taking any $c_n \models q|_{b_{\leq n} b'_{\leq n} c_{<n}}$ we obtain the desired sequence.

Now by Ramsey, passing to an infinite subsequence we may assume that either $\models \phi(c_i,b_j)$ for all $i<j$, or $\models \neg\phi(c_i,b_j)$ for all $i<j$. In the first case, the sequence $(c_i,b'_i)_{i\in\omega}$ witnesses that $\phi(x,y)$ is not stable, in the second case the sequence $(c_i,b_{i+1})_{i\in\omega}$ witnesses this. □

Corollary 3.12. *If φ is stable, then for any set A we have $|S_\varphi(A)| \leq |A| + \aleph_0$.*

Proof. Let $\mathcal{L}'$ be a countable reduct of $\mathcal{L}$ containing φ. Then given any set A, by Lowenhein-Skolem there is a model containing it of size $|A| + \aleph_0$. By 3.11 every φ type over M is definable, and there are at most $|M| + |\mathcal{L}'|$ possible definitions (and every type over A extends to some type over M by compactness). □

Definition 3.13. We define *Shelah's local* 2*-rank* taking values in $\{-\infty\} \cup \omega \cup \{+\infty\}$ by induction on $n \in \omega$ (there are many other related ranks). Let Δ be a set of L-formulas, and $\theta(x)$ a partial type over $\mathbb{M}$.

- $R_\Delta(\theta(x)) \geq 0$ iff $\theta(x)$ is consistent (and $-\infty$ otherwise).
- $R_\Delta(\theta(x)) \geq n+1$ if for some $\phi(x,y) \in \Delta$ and $a \in \mathbb{M}_y$ we have both $R_\Delta(\theta(x) \wedge \phi(x,a)) \geq n$ and $R_\Delta(\theta(x) \wedge \neg\phi(x,a)) \geq n$.
- $R_\Delta(\theta(x)) = n$ if $R_\Delta(\theta(x)) \geq n$ and$R_\Delta(\theta(x)) \not\geq n+1$, and $R_\Delta(\theta(x)) = \infty$ if $R_\Delta(\theta(x)) \geq n$ for all $n \in \omega$.

If $\phi(x,y)$ is a formula, we write R_ϕ instead of $R_{\{\phi\}}$.

Proposition 3.14. *$\phi(x,y)$ is stable if and only if $R_\phi(x=x)$ is finite (and so also $R_\phi(\theta(x))$ is finite for any partial type θ). Here $x = (x_i : i \in I)$ is a tuple of variables, and $x = x$ is an abuse of notation for $\bigwedge_{i \in I} x_i = x_i$.*

Proof. Assume that $\phi(x,y)$ is unstable, i.e. it has the k-order property for all $k \in \omega$. By compactness we find $(a_i b_i : i \in [0,1])$ such that $\models \phi(a_{i,} b_j) \iff i < j$. We know that both $\phi\left(x, b_{\frac{1}{2}}\right)$ and $\neg\phi\left(x, b_{\frac{1}{2}}\right)$ contain dense subsequences of a_i's. Each of these sets can be split again, by $\phi\left(x, b_{\frac{1}{4}}\right)$ and $\phi\left(x, b_{\frac{3}{4}}\right)$, resp., etc.

Conversely, assume that the rank is infinite, then we can find an infinite tree of parameters $B = (B_\eta : \eta \in 2^{<\omega})$ such that for every $\eta \in 2^\omega$ the set of formulas $\left\{\phi^{\eta(i)}\left(x, b_{\eta|i}\right) : i < \omega\right\}$ is consistent (rank being $\geq k$ guarantees that we can find such a tree of height k, and then use compactness to find one of infinite height). This gives us that $|S_\phi(B)| > |B|$, which by Corollary 3.12 implies that $\phi(x,y)$ is unstable. □

3.3. **Erdős-Hajnal property and coarse dimension.** See https://arxiv.org/abs/1504.08252.

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