MATH 115A (CHERNIKOV), SPRING 2017

PROBLEM SET 3

DUE THURSDAY, APRIL 27

Problem 1. Do Exercise 1, Section 1.6. Justify each answer!

Problem 2. Determine which of the following sets are bases for $P_2(\mathbb{R})$ and justify your answer:

(1) $\{5+12x, 6-x, 3+18x\}$.
(2) $\left\{1+2x+x^2, 3+x^2, x+x^2\right\}$.
(3) $\left\{1-2x-2x^2, -2+3x-x^2, 1-x+6x^2\right\}$.
(4) $\left\{1+2x-x^2, 4-2x+x^2, -1+18x-9x\right\}$.
(5) $\left\{1+2x+x^2, 1-x+6x^2, 3-4x-10x^2, 16x^2\right\}$.

Problem 3. The set of solutions to the system of linear equations

$$x_1-2x_2+x_3=0$$

$$2x_1-3x_2+x_3=0$$

is a subspace of $\mathbb{R}^3$. Find a basis for this subspace.

Problem 4. Suppose that V is a vector space of dimension n, and let W be a subspace of V of dimension m (so $m \leq n$). Show that for every integer k such that $m \leq k \leq n$ there is a subspace U of V such that $W \subseteq U \subseteq V$ and $\dim(U)=k$.

Problem 5. Let $T:\mathbb{R}^4 \to \mathbb{R}^3$ be a linear map with
$T(1,0,0,0)=(0,1,2)$,
$T(0,1,0,0)=(0,1,2)$,
$T(0,0,1,0)=(1,0,2)$,
$T(0,0,0,1)=(1,1,4)$.
Determine a basis for the range $R(T)$ of T, a basis for the null space $N(T)$ of T, and compute the dimension of $N(T)$.

Problem 6. Is there a linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^2$ such that $T(1,0,-2)=(1,1)$ and $T(-2,0,4)=(2,6)$?

Problem 7. We consider $V=M_{n\times n}(\mathbb{R})$. The *trace* of an $n\times n$ matrix $A \in M_{n\times n}(\mathbb{R})$, denoted by $\operatorname{tr}(A)$, is the sum of the diagonal entries of A:

$$\operatorname{tr}(A)=A_{11}+A_{22}+\ldots+A_{nn}.$$

Consider the map $T: M_{n\times n}(\mathbb{R}) \to \mathbb{R}$ defined by $T(A)=\operatorname{tr}(A)$.

Show that it is a linear transformation, and determine the null space $N(T)$ of T and the dimension of $N(T)$.

Problem 8. Let $v_1, \ldots, v_n$ be vectors in a vector space V over a field F. Consider the map

$$T : F^n \to V, (a_1, \ldots, a_n) \mapsto a_1 v_1 + \ldots + a_n v_n.$$

Show that T is linear, and moreover:

(1) T is injective if and only if $\{v_1, \ldots, v_n\}$ is linearly independent.
(2) T is surjective if and only if $\{v_1, \ldots, v_n\}$ generates V.
(3) T is bijective if and only if $\{v_1, \ldots, v_n\}$ is a basis for V.