MATH 115A (CHERNIKOV), SPRING 2017

PROBLEM SET 2

DUE THURSDAY, APRIL 20

Problem 1. Let V be a vector space, v a vector in V and $S \subseteq V$. In each of the following cases, determine whether $v \in \operatorname{Span}(S)$ (and justify).

(1) $V = \mathbb{R}^3$, $v = (-1, 2, 1)$, $S = \{(1, 0, 2), (-1, 1, 1)\}$,
(2) $V = P_6(\mathbb{R})$, $v = -x^3 + 2x^2 + 3x + 3$, $S = \{x^3 + x^2 + x + 1, x^2 + x + 1, x + 1\}$,
(3) $V = M_{2\times 2}(\mathbb{R})$, $v = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $S = \left\{ \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$.

Problem 2. Let S_1 and S_2 be subsets of a vector space V.

(1) Prove that $\operatorname{Span}(S_1 \cap S_2) \subseteq \operatorname{Span}(S_1) \cap \operatorname{Span}(S_2)$.
(2) Give an example in which $\operatorname{Span}(S_1 \cap S_2)$ and $\operatorname{Span}(S_1) \cap \operatorname{Span}(S_2)$ are equal, and one in which they are unequal.

Problem 3. Let V be a vector space over a field F.

(1) Prove that for any vector $v \in V$, $\operatorname{Span}(v) = \{av : a \in F\}$. What does this mean geometrically in $\mathbb{R}^3$?
(2) Let $u \neq v$ be two vectors in a vector space V over a field F. Show that the set $\{u, v\}$ is linearly dependent if and only if u is a multiple of v, or v is a multiple of u.

Problem 4. Show that a subset W of a vector space V is a subspace if and only if $\operatorname{Span}(W) = W$.

Problem 5. Let $M_{m\times n}(\mathbb{R})$ be the vector space of all m-by-n matrices with real entries.

For an $m\times n$ matrix $A \in M_{m\times n}(\mathbb{R})$, its *transpose* A^t is the $n\times m$ matrix obtained from A by interchanging the rows with the columns. That is, $(A^t)_{ij} = A_{ji}$ for all $1 \leq i \leq m, 1 \leq j \leq n$. So for example if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^t = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$.

A *symmetric* matrix is a matrix A such that $A^t = A$ (so it has to be a square matrix, that is $m = n$).

Let W be the set of all symmetric matrices in $M_{2\times 2}(\mathbb{R})$.

(1) Show that W is a subspace of $M_{2\times 2}(\mathbb{R})$ (Hint: you will need to prove that $(aA + bB)^t = aA^t + bB^t$ for any $A, B \in M_{2\times 2}(\mathbb{R})$ and $a, b \in \mathbb{R}$).
(2) Let
$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$
Show that $\operatorname{Span}(\{A_1, A_2, A_3\}) = W$.

Problem 6. Consider the following sets of vectors.

(1) $\{(1,0,0),(1,1,1)\}$ in $\mathbb{R}^3$,
(2) $\left\{\begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}, \begin{pmatrix} -2 & 6 \\ 4 & -8 \end{pmatrix}\right\}$ in $M_{2\times 2}(\mathbb{R})$,
(3) $\{(s,-r,0),(t,0,r),(0,t,s)\}$, where $r,s,t \in \mathbb{R}$.

Determine if they are linearly dependent or linearly independent (and justify, with case distinctions in (3) if necessary).

Problem 7. Let $V = \mathbb{R}^3$. Find three vectors $w,v,z \in V$ with the following properties:

(1) $\operatorname{Span}(\{w,v\}) = \operatorname{Span}(\{v,z\}) = \operatorname{Span}(\{w,v,z\})$,
(2) $\operatorname{Span}(\{w,v,z\}) \neq \operatorname{Span}\{w,z\}$.

Suppose that w,v,z are **any** three vectors in **any** vector space V with the above listed properties. Prove or disprove each of the following statements:

(1) w,v are linearly independent.
(2) v,z are linearly independent.
(3) w,z are linearly independent.

Problem 8. Determine which of the following sets are bases for $\mathbb{R}^3$:

(1) $\{(1,2,-1),(1,0,2),(2,1,1)\}$,
(2) $\{(2,-4,1),(0,3,-1),(6,0,-1)\}$,
(3) $\{(-1,3,1),(2,-4,-3),(-3,8,2)\}$,
(4) $\{(1,0,-1),(2,5,1),(0,-4,3),(7,2,2)\}$.