Midterm 1 Review Sheet

Vector spaces

Definition. A vector space \( V \) over a field \( F \) is a set with two operations, addition and scalar multiplication, (so for any \( x, y \in V \) and \( a \in F \), \( x + y \) and \( ax \) are in \( V \)) such that the following conditions hold:

1. \( x + y = y + x \) for all \( x, y \in V \) (commutativity)
2. \( (x + y) + z = x + (y + z) \) for all \( x, y, z \in V \) (associativity)
3. There exists an element 0 in \( V \) such that \( x + 0 = x \) for all \( x \in V \) (identity)
4. For each \( x \in V \) there is an element \( y \) in \( V \) such that \( x + y = 0 \) (\( y \) is an inverse of \( x \))
5. \( 1 \cdot x = x \) for all \( x \in V \) (where 1 is the multiplicative identity of \( F \))
6. \( a(bx) = (ab)x \) for all \( a, b \in F \) and \( x \in V \)
7. \( a(x + y) = ax + ay \) for all \( a \in F \) and \( x, y \in V \)
8. \( (a + b)x = ax + bx \) for all \( a, b \in F \) and \( x \in V \)

Elements of \( V \) are called vectors.
Elements of \( F \) are called scalars.

Theorem 1 (Cancellation law)

Let \( V \) be a vs. and let \( x, y, z \in V \).

If \( x + z = y + z \), then \( x = y \).

Corollary
1) In any vector space \( V \), there is a unique element 0 satisfying (1) — the zero vector of \( V \).
2) For any vs. \( V \) and any \( x \in V \), there is a unique element \( y \) in \( V \) satisfying (1).
It is called the inverse of \( x \) and denoted by \(-x\).

Theorem 12
Let \( V \) be a vs. over \( F \).

For all \( x \in V \) and \( a \in F \) we have:

1) \( 0 \cdot x = 0 \) (Note: the 1st 0 is a scalar in \( F \), the 2nd one is the zero vector in \( V \)).
2) \( (-a) \cdot x = -(a \cdot x) \)
3) \( a \cdot 0 = 0 \) (Note: this is the zero vector of \( V \) on both sides).

Subspaces

Definition. Let \( V \) be a vs. A subset \( W \subseteq V \) is a subspace of \( V \) if \( W \) itself is a vs. with respect to the addition and scalar multiplication defined on \( V \).

Theorem 13
Let \( V \) be a vs., and let \( W \subseteq V \) be a subset of \( V \).

Then \( W \) is a subspace of \( V \) if and only if all of the following conditions hold:

(a) \( 0 \in W \)
(b) \( x + y \in W \) for all \( x, y \in W \) (\( W \) is closed under addition)
(c) \( c \cdot x \in W \) for all \( c \in F \) and \( x \in W \) (\( W \) is closed under scalar multiplication).

Theorem 14
Let \( V \) be a vs. over \( F \).

If \( W_1, \ldots, W_n \) are subspaces of \( V \), then the set \( W = W_1 \cap W_2 \cap \cdots \cap W_n \) is also a subspace of \( V \).

Linear combinations

Definition. Let \( V \) be a vs., and let \( S \subseteq V \) be a non-empty subset of \( V \).

A vector \( v \) in \( V \) is a linear combination of \( S \) if one can write

\[ v = a_1 u_1 + a_2 u_2 + \cdots + a_n u_n \]

for some vectors \( u_1, \ldots, u_n \) in \( S \) and some scalars \( a_1, \ldots, a_n \) in \( F \).
2) The span of $S$, denoted $\text{Span}(S)$, is the subset of $V$ consisting precisely of all linear combinations of $S$. That is, 
\[ \text{Span}(S) = \{ \sum_{i=1}^{n} a_i u_i : n \in \mathbb{N}, a_i \in F, u_i \in S \} \]
For convenience, we define $\text{Span}(\emptyset) = \{0\}$.

**Theorem 1.5.** Let $S$ be any subset of a v.s. $V$. Then:
1) $\text{Span}(S)$ is a subset of $V$.
2) Any subset of $V$ that contains $S$ must also contain $\text{Span}(S)$.

**Definition.** Let $V$ be a v.s. and $S$ a subset of $V$.
We say that $S$ generates (or spans) $V$ if $\text{Span}(S) = V$.

**Definition.** A subset $S$ of a v.s. $V$ is linearly dependent if there exist a finite number of distinct vectors $u_1, \ldots, u_m$ in $S$ and scalars $a_1, \ldots, a_m \in F$, with at least one $a_i \neq 0$, such that
\[ a_1 u_1 + \cdots + a_m u_m = 0. \]
We say $S$ is linearly independent if it is not linearly dependent.

**Theorem 1.6.** Let $V$ be a v.s. and $S, S_2 \subseteq V$ be two subsets of $V$.
1) If $S_2$ is linear dependent, then $S \cup S_2$ is also linearly dependent.
2) If $S_2$ is linear independent, then $S$ is also linear independent.

**Theorem 1.7.** Let $S$ be a linearly independent subset of a vector space $V$.
Let $V$ be any vector in $V$ not contained in $S$.
Then $S \cup \{v\}$ is linearly independent if and only if $v \in \text{Span}(S)$.

**Bases and dimension.**

**Definition.** A **basis** for a v.s. $V$ is a subset of $V$ which is linearly independent and generates $V$.

**Theorem 1.8.** A subset $\{u_1, \ldots, u_n\}$ of a v.s. $V$ is a basis if and only if every vector $v \in V$ can be written uniquely in the form
\[ v = a_1 u_1 + \cdots + a_n u_n, \]
where $a_i \in F$.
(“uniquely” here means that there is only one possible choice of the scalars $a_i, \ldots, a_n \in F$ satisfying the equality)

**Theorem 1.9.** If $v \in \text{span}(S)$ is generated by a finite subset $S$, then some subset of $S$ is a basis.
It follows that every finitely generated v.s. has a basis.

**Theorem 1.10.** (Replacement Theorem)
Let $V$ be a v.s. generated by a set $G \subseteq V$ with $|G| = n$, and let $L$ be a linearly independent subset of $V$.
Then $m \leq n$, and there exists $H \subseteq G$ with $|H| = m$ such that $L \cup H$ generates $V$.

**Corollary 1.** Let $V$ be a finitely generated v.s. Then every basis for $V$ has the same number of elements.

**Definition.** A v.s. $V$ is finite-dimensional if it has a finite basis.

The (unique) number of vectors in a basis for $V$ is called the **dimension** of $V$, denoted $\dim(V)$.

If there is no finite basis, then $V$ is **infinite-dimensional**.

**Corollary 2.** Let $V$ be a v.s. of dimension $n$. Then:
a) Any generating set for $V$ must contain at least $n$ vectors.

b) Any linearly independent subset of $V$ with $n$ elements is a basis.

c) Every linearly independent subset of $V$ can be extended to a basis for $V$.

**Theorem 1.3** Let $W$ be a subspace of a vector space $V$ with $\dim(V) \geq n$.

Then $\dim(W) \leq \dim(V)$.

Moreover, if $\dim(W) = \dim(V)$, then $V = W$. 