

PROBLEM SET 9

DUE FRIDAY, MAY 27

Problem 1. Do Exercise 1, Section 6.1. Justify each answer.

Problem 2. Show that each of the following is not an inner product on the given vector spaces.

- (1) $\langle (a, b), (c, d) \rangle = ac - bd$ on \mathbb{R}^2 .
- (2) $\langle A, B \rangle = \text{tr}(A + B)$ on $M_{2 \times 2}(\mathbb{R})$.
- (3) $\langle f(x), g(x) \rangle = \int_0^1 f'(t)g(t) dt$ on $P(\mathbb{R})$, where $'$ denotes differentiation.

Problem 3. Let V be an inner product space, and suppose that x and y are orthogonal vectors in V . Prove that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$. Deduce the Pythagorean theorem in \mathbb{R}^2 from it.

Problem 4. Let T be a linear operator on an inner product space V , and suppose that $\|T(x)\| = \|x\|$ for all x . Prove that T is injective.

Problem 5. Do Exercise 1, Section 6.2. Justify each answer.

Problem 6. Apply the Gram-Schmidt process to the given subset S of the inner product space V to obtain an orthogonal basis for $\text{Span}(S)$. Then normalize the vectors in this basis to obtain an orthonormal basis β for $\text{Span}(S)$.

- (1) $V = \mathbb{R}^3$ with the dot product and $S = \{(1, 0, 1), (0, 1, 1), (1, 3, 3)\}$.
- (2) $V = P_2(\mathbb{R})$ with the inner product $\langle f(x), g(x) \rangle = \int_0^1 f(t)g(t) dt$ and $S = \{1, x, x^2\}$.
- (3) $V = M_{2 \times 2}(\mathbb{R})$ with the Frobenius inner product and

$$S = \left\{ \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix}, \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix} \right\}.$$

Problem 7. Let β be a basis for a subspace W of an inner product space V , and let $z \in V$. Prove that $z \in W^\perp$ if and only if $\langle z, v \rangle = 0$ for every $v \in \beta$.

Problem 8. Let V be an inner product space, let W be a finite dimensional subspace of V and let S, S_0 be subsets of V . Prove the following results.

- (1) If $x \notin W$, prove that there exists $y \in V$ such that $y \in W^\perp$, but $\langle x, y \rangle \neq 0$ (Hint: use Theorem 6.6).
- (2) $S_0 \subseteq S$ implies that $S^\perp \subseteq S_0^\perp$.
- (3) $S \subseteq (S^\perp)^\perp$ and $\text{Span}(S) \subseteq (S^\perp)^\perp$.
- (4) $W = (W^\perp)^\perp$ (Hint: use (1)).