Classical (discrete) case

Def. An $\mathcal{L}$-structure $\mathcal{M}$ is <u>$\aleph_0$-categorical</u> if it is the unique countable model of its first-order theory $Th(\mathcal{M})$.

Fact (Ryll-Nardzewski). A ctble structure $\mathcal{M}$ is $\aleph_0$-cat iff the action $Aut(\mathcal{M}) \curvearrowright \mathcal{M}$ is oligomorphic.
I.e., the diagonal action $Aut(\mathcal{M}) \curvearrowright \mathcal{M}^n$, $\sigma\cdot(a_1,\dots,a_n) = (\sigma(a_1),\dots,\sigma(a_n))$, has $<\infty$ orbits $\forall n$.

Main examples of $\aleph_0$-cat structures are given by:

Def $\mathcal{M}$ is homogeneous if every isomorphism $f: A \to B$ with A, B finitely generated substruct. of $\mathcal{M}$, extends to some $f' \in Aut(\mathcal{M})$.

Rem A homogeneous $\mathcal{M}$ is $\aleph_0$-cat $\Leftrightarrow$ $|\{A/\cong \,:\, A \leq M \text{ gen. by } n \text{ elements}\}| < \infty$ for all n.

Ex $(\mathbb{Q}, <)$, the countable random graph, the ctbl. atomless Bool. algebra, [$\aleph_0$-dim. vec. space over $\mathbb{F}_p$ is $\aleph_0$-cat, but not homog].

For such $\mathcal{M}$, $Aut(\mathcal{M})$ is a permutation group, i.e. a closed subgroup of S_∞ (the group of all permutations of $\mathbb{N}$). (Pointwise convergence top, basis of neighb. of the 1 given by $Stab(A)$, $A \subseteq \mathbb{N}$ finite.

Given a perm. group $G \overset{Aut(\mathcal{M})}{=}$ on M, can recover M up to interdefinability (i.e. find a structure on the same set, possibly in a different lang, with the same definable relations) — just add a predicate for every orbit on M^n.
What if G is only given as a top. group?

Def 1) An interpr. of $\underset{\mathcal{L}}{\mathcal{M}}$ in $\underset{\mathcal{L}'}{\mathcal{N}}$ is given by a surj. map f from a subset of N^n onto M s.t. $\forall$ def. rel. $X \subseteq M^k$, its preimage $f^{-1}(X)$ is def. in N (in part. $f^{-1}(M)$ is def., and "=" on M).
2) M, N are bi-interpr. if $\exists$ an interpr. of $\mathcal{M}$ in N and of N in M s.t. the composit. interpr. of $\mathcal{M}$ in itself is def.$^{(\text{in } M)}$ isomorphic to M. (and likewise for N).

Fact (Ahlbrandt, Ziegler)
1) Let M, N be ω-cat. Then M and N are bi-interpr. iff $Aut(M)$ and $Aut(N)$ are isomorph. as top. groups.

Model theory: studies properties of theories up to bi-interpr.
Hence, a <u>dictionary</u> between model-theoretic prop. of $\mathcal{M}$ and top. prop. of $Aut(\mathcal{M})$ can be developed.
stability.
Note: DLO, Rand graph not stable, but vect. space is stable.

Cont. struct: Complete metric space, all relations are unif. cont.
Cont. logic: $\aleph_0$-cat $\rightsquigarrow$ separably cat.
...

Now if M is sep. cat, $Aut(M)$ is a Polish group. (with the top. of pointwise convergence).

Fact (Tsankov, BY) Aut. groups of sep. cat. structures are precisely the Roelcke precompact Polish gps.
(A top. gp. G is Roelcke precompact if $\forall$ neighb. $U \ni 1_G$, $\exists F \subseteq_{fin} G$ s.t. $UFU = G$.

<u>Ex</u>: - sep, inf. dim. Hilb space
- the measure algebra of a stand. prob. space
- sep. atomless L^p Banach lattices ($p<\infty$).

New examples in top. dynamics — $Aut(\mathcal{M})$ for $\mathcal{M}$ produced by Hrushovski construction.