Model theoretic approach to de Finetti theory

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Interactions between Logic, Topological structures and Banach spaces theory

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Model theory

- We fix a complete countable first-order theory T in a language L.
- Let $\mathbb{M}$ be a monster model of T (i.e. κ^*-saturated and κ^*-homogeneous for some sufficiently large cardinal κ^*).
- Given a set $A \subseteq \mathbb{M}$, we let $S(A)$ denote the space of types over A (i.e. the Stone space of ultrafilters on the Boolean algebra of A-definable subsets of $\mathbb{M}$).

Shelah's classification

- Morley's theorem: for a countable T, if it has only one model of some uncountable cardinality (up to isomorphism), then it has only one model of every uncountable cardinality.
- Morley's conjecture: for a countable theory T, the number of its models of size κ is non-decreasing on uncountable κ.
- In his work on Morley's conjecture, Shelah had isolated an important class of stable theories and had developed a lot of machinery to analyze types and models of stable theories.

Stability

Definition

1. We say that T encodes a linear order if there is a formula $\phi(\bar{x}, \bar{y}) \in L$ and $(\bar{a}_i : i \in \omega)$ in $\mathbb{M}$ such that $\mathbb{M} \models \phi(\bar{a}_i, \bar{a}_j) \Leftrightarrow i < j$.
2. A theory T is *stable* if it cannot encode a linear order.

- Examples of first-order theories: equivalence relations, modules, algebraically closed fields, separably closed fields (Wood), free groups (Sela), planar graphs (Podewski and Ziegler).

Stability: number of types

Fact

The following are equivalent:

1. *T is stable.*
2. *For some cardinal κ we have* $\sup\{|S(M)| : M \models T, |M| = \kappa\} = \kappa$.
3. *For every cardinal κ we have* $\sup\{|S(M)| : M \models T, |M| = \kappa\} \leq \kappa^{|T|}$.

Stability: indiscernible sequences

Definition

1. $(a_i : i \in \omega)$ is an *indiscernible sequence* over a set B if $\operatorname{tp}(a_{i_0} \dots a_{i_n}/B) = \operatorname{tp}(a_{j_0} \dots a_{j_n}/B)$ for any $i_0 < \dots < i_n$ and $j_0 < \dots < j_n$ from ω.
2. $(a_i : i \in \omega)$ is an *indiscernible set* over B if $\operatorname{tp}(a_{i_0} \dots a_{i_n}/B) = \operatorname{tp}\left(a_{\sigma(i_0)} \dots a_{\sigma(i_n)}/B\right)$ for any $\sigma \in S_\infty$.

Fact

The following are equivalent:

1. *T is stable.*
2. *Every indiscernible sequence is an indiscernible set.*

Stability: the independence relation

Fact

The following are equivalent:

1. *T is stable.*
2. *There is an independence relation $\mathop{\smile\!\!\!\!\!\!\downarrow}$ on small subsets of $\mathbb{M}$ (i.e. of cardinality $< \kappa^*$) satisfying the following axioms:*
   - *Invariance:* $A \mathop{\smile\!\!\!\!\!\!\downarrow}_C B, \sigma \in \mathrm{Aut}(\mathbb{M}) \Rightarrow \sigma(A) \mathop{\smile\!\!\!\!\!\!\downarrow}_{\sigma(C)} \sigma(B)$.
   - *Symmetry:* $A \mathop{\smile\!\!\!\!\!\!\downarrow}_C B \Leftrightarrow B \mathop{\smile\!\!\!\!\!\!\downarrow}_C A$.
   - *Monotonicity:* $A \mathop{\smile\!\!\!\!\!\!\downarrow}_C B, A' \subseteq A, B' \subseteq B \Rightarrow A' \mathop{\smile\!\!\!\!\!\!\downarrow}_C B'$.
   - *Base monotonicity:* $A \mathop{\smile\!\!\!\!\!\!\downarrow}_D BC \Rightarrow A \mathop{\smile\!\!\!\!\!\!\downarrow}_{DC} B$.
   - *Transitivity:* $A \mathop{\smile\!\!\!\!\!\!\downarrow}_{CD} B, A \mathop{\smile\!\!\!\!\!\!\downarrow}_D C \Rightarrow A \mathop{\smile\!\!\!\!\!\!\downarrow}_D BC$.
   - *Extension:* $A \mathop{\smile\!\!\!\!\!\!\downarrow}_C B, D \supseteq B \Rightarrow \exists A'$ *such that* $\mathrm{tp}(A'/BC) = \mathrm{tp}(A/BC)$ *and* $A' \mathop{\smile\!\!\!\!\!\!\downarrow}_C D$.
   - *Boundedness: For every* $B \supseteq C$ *and finite* n *we have* $\left|\left\{\mathrm{tp}(A/B) : |A| = n, A \mathop{\smile\!\!\!\!\!\!\downarrow}_C B\right\}\right| \leq 2^{|C|}$.
   - *Finite character:* $A' \mathop{\smile\!\!\!\!\!\!\downarrow}_C B$ *for all finite* $A' \subseteq A \Rightarrow A \mathop{\smile\!\!\!\!\!\!\downarrow}_C B$.
   - *Local character: For every finite* A *and any* B*, there is some* $C \subseteq B, |C| \leq |T|$ *such that* $A \mathop{\smile\!\!\!\!\!\!\downarrow}_C B$.

Stability: independence relation

- In fact, if such a relation exists then it has to come from Shelah's *non-forking* — a canonically defined way of producing "generic" extensions of types.
- Examples: linear independence in vector spaces, algebraic independence in algebraically closed fields.

Stability tools

Definition

A sequence $(a_i)_{i \in \omega}$ in $\mathbb{M}$ is a *Morley sequence* in a type $p \in S(B)$ if it is a sequence of realizations of p indiscernible over B and such that moreover $a_i \mathop{\smile\!\!\!\!\!\!|}_B a_{<i}$ for all $i \in \omega$.

Fact

In a stable theory, every type admits a Morley sequence (Erdős-Rado + compactness + properties of forking independence).

- An important technical tool in the development of stability.
- Example: an infinite basis in a vector space.

Stability tools

Definition

In a stable theory, every stationary type has a *canonical base* — a small set such that every automorphism of $\mathbb{M}$ fixing it fixes the global non-forking extension of p.

- If we want every type to have a canonical base, we might have to add imaginary elements for classes of definable equivalence relations to the structure, but this is a tame procedure.
- The *definable closure* of a set $A \subseteq \mathbb{M}$: $\mathrm{dcl}(A) = \{b \in \mathbb{M} : \exists \phi(x) \in L(A) \text{ s.t. } \models \phi(b) \wedge |\phi(\mathbb{M}, b)| = 1\}$.

Theorem

(Folklore) Every indiscernible sequence $(a_i)_{i \in \omega}$ is Morley over the canonical base of the limit type, which is equal to $\bigcap_{n \in \omega} \mathrm{dcl}^{\mathrm{eq}}(a_{\geq n})$.

Exchangeable sequences of random variables

- Let $(\Omega, \mathcal{F}, \mu)$ be a probability space.
- Let $\bar{X} = (X_i)_{i \in \omega}$ be a sequence of $[0,1]$-valued random variables on Ω.
- $\bar{X}$ is *exchangeable* if $(X_{i_0}, \ldots, X_{i_n}) \stackrel{d}{=} (X_0, \ldots, X_n)$ for any $i_0 \neq \ldots \neq i_n$ and $n \in \omega$.
- Example: A sequence of i.i.d. (independent, identically distributed) random variables.
- Question: Is the converse true?

Classical de Finetti's theorem

Fact

A sequence of random variables $(X_i)_{i\in\omega}$ *is exchangeable if and only if it is i.i.d. over its tail* σ*-algebra* $T = \bigcap_{n\in\omega} \sigma(X_{\geq n})$.

Continuous logic

- Reference: Ben Yaacov, Berenstein, Henson, Usvyatsov "Model theory for metric structures".
- Every structure M is a complete metric space of bounded diameter, with metric d.
- Signature:
  - function symbols with given moduli of uniform continuity (correspond to uniformly continuous functions from M^n to M),
  - predicate symbols with given moduli of uniform continuity (uniformly continuous functions from M to $[0,1]$).
- Connectives: the set of all continuous functions from $[0,1] \to [0,1]$, or any subfamily which generates a dense subset (e.g. $\left\{\neg, \frac{x}{2}, \dot{-}\right\}$).
- Quantifiers: sup for $\forall$, inf for $\exists$.
- This logic admits a compactness theorem, etc.

Stability in continuous logic

- Summary: everything is essentially the same as in the classical case (Ben Yaacov, Usvyatsov "Continuous first-order logic and local stability").
- Of course, modulo some natural changes: cardinality is replaced by the density character, in acl "finite" is replaced by "compact", some equivalences are replaced by the ability to approximate uniformly, etc.

The theory of random variables

- Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, and let $L^1((\Omega, \mathcal{F}; \mu), [0,1])$ be the space of $[0,1]$-valued random variables on it.
- We consider it as a continuous structure in the language $L_{\mathrm{RV}} = \left\{0, \neg, \frac{x}{2}, \dot{-}\right\}$ with the natural interpretation of the connectives (e.g. $\left(X \dot{-} Y\right)(\omega) = X(\omega) \dot{-} Y(\omega)$) and the distance $d(X, Y) = \mathbf{E}[|X - Y|] = \int_\Omega |X - Y| \, d\mu$.

The theory of random variables

- Consider the following continuous theory RV in the language L_{RV}, we write 1 as an abbreviation for $\neg 0$, $E(x)$ for $d(0,x)$ and $x \wedge y$ for $x \dot{-} \left(x \dot{-} y\right)$:
  - $E(x) = E\left(x \dot{-} y\right) + E(y \wedge x)$
  - $E(1) = 1$
  - $d(x,y) = E\left(x \dot{-} y\right) + E\left(y \dot{-} x\right)$
  - $\tau = 0$ for every term τ which can be deduced in the propositional continuous logic.
- The theory ARV is defined by adding:
  - Atomlessness: $\inf_y \left(E(y \wedge \neg y) \vee \left| E(y \wedge x) - \frac{E(x)}{2} \right|\right) = 0.$

The theory of random variables: basic properties

Definition

Let $\sigma(A) \subseteq \mathcal{F}$ denote the minimal complete subalgebra with respect to which every $X \in A$ is measurable.

Fact

[Ben Yaacov, "On theories of random variables"]

1. $M \models \mathrm{RV} \Leftrightarrow$ *it is isomorphic to* $L^1(\Omega, [0,1])$ *for some probability space* $(\Omega, \mathcal{F}, \mu)$.
2. $M \models \mathrm{ARV} \Leftrightarrow$ *it is isomorphic* $L^1(\Omega, [0,1])$ *for some atomless probability space* $(\Omega, \mathcal{F}, \mu)$.
3. ARV *is the model completion of the universal theory* RV *(so every probability space embeds into a model of* ARV*).*
4. ARV *eliminates quantifiers, and two tuples have the same type over a set* $A \subseteq M$ *if and only if they have the same joint conditional distribution as random variables over* $\sigma(A)$.

The theory of random variables: stability

Fact

[Ben Yaacov, "On theories of random variables"]

1. ARV *is* $\aleph_0$*-categorical (i.e., there is a unique separable model) and complete.*
2. ARV *is stable (and in fact* $\aleph_0$*-stable).*
3. ARV *eliminates imaginaries.*
4. *If* $M \models \mathrm{ARV}$ *and* $A \subseteq M$*, then* $\mathrm{dcl}(A) = \mathrm{acl}(A) = L^1(\sigma(A), [0,1]) \subseteq M$.
5. *Model-theoretic independence coincides with probabilistic independence:* $A \mathop{\smile\!\!\!\!\!\!|}_B C \Leftrightarrow \mathbb{P}[X|\sigma(BC)] = \mathbb{P}[X|\sigma(B)]$ *for every* $X \in \sigma(A)$*. Moreover, every type is stationary.*

Back to de Finetti

- As every model of RV embeds into a model of ARV, wlog our sequence of random variables is from $\mathbb{M} \models$ ARV.
- Recall: In a stable theory, every indiscernible sequence is an indiscernible set.

Corollary

(Ryll-Nardzewski) A sequence of random variables is exchangeable iff it is contractable (i.e. $X_{i_0} \dots X_{i_n} \stackrel{d}{=} X_0 \dots X_n$).

- Recall: In a stable theory, every indiscernible sequence is a Morley sequence over the definable tail closure.

Corollary

De Finetti's theorem.

Multi-dimensional de Finetti

- A reformulation of de Finetti's theorem:

Fact

$(X_i)_{i\in\omega}$ *is exchangeable iff there is a measurable function* $f:[0,1]^2\to\Omega$ *and some i.i.d.* $U(0,1)$ *random variables* α *and* $(\xi_i)_{i\in\omega}$ *such that a.s.* $X_n = f(\alpha,\xi_i)$.

- f is not unique here, and we might have to extend the basic probability space.

Multi-dimensional de Finetti

- So, 1-dimensional case was already folklore in stability theory.
- Multi-dimensional case, exchangeable arrays:

Fact

[Aldous, Hoover] An array of random variables $X = (X_{i,j})$ is exchangeable iff there exist a measurable function $f : [0,1]^4 \to \Omega$ and some i.i.d. $U(0,1)$ random variables $\alpha, \xi_i, \eta_j, \zeta_{i,j}$ such that a.s. $X_{i,j} = f(\alpha, \xi_i, \eta_j, \zeta_{i,j})$.

- Kallenberg for *n*-dimensional case.
- Again, an exercise in forking calculus gives the required statement about probabilistic independence, and basic model theoretic properties of ARV allow to conclude.

Directions

- Keisler randomization: instead of sampling from $[0,1]$ we can be sampling from a model of an arbitrary stable (continuous) first-order theory (ARV can be viewed as a "randomization of the equality") .
- Non-commutative probability theory: no longer stable, no model complete theory and no quantifier elimination, but there is an appropriate notion of independence on quantifier-free types.