What is... Fraissé construction?

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”What is...” seminar at FU Berlin,
30 January 2009

Let M be some countable structure in a fixed language (graph, group, linear order or whatever).

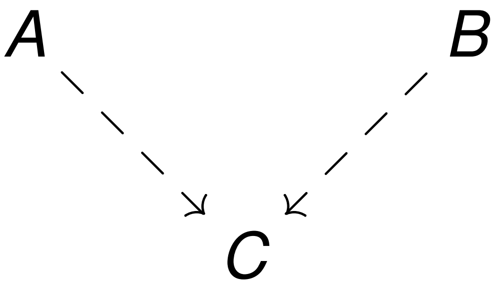
Definition: Let $\mathcal{C}(M)$ be the category of substructures of M and $\mathcal{C}_0(M)$ its full subcategory of finitely generated substructures.

Question: When can we recover the structure M from $\mathcal{C}_0(M)$ alone?

Properties of $\mathcal{C}_0(M)$

Joint Embedding Property (JEP)

For any $A, B \in \mathcal{C}_0(M)$ there is some $C \in \mathcal{C}_0(M)$ such that



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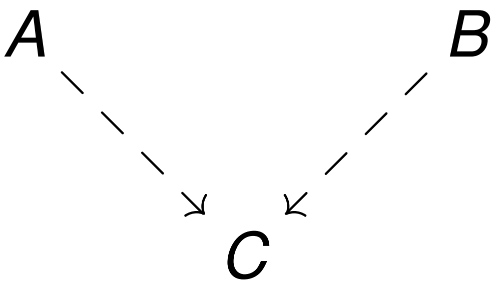

Hereditary Property (HP)

If A is in $\mathcal{C}_0(M)$ and $B \in \mathcal{C}(M)$ embeds $B \to A$, then $B \in \mathcal{C}_0(M)$.

Reconstruction: first attempt

Observation: For every countable category of finitely generated structures $\mathcal{C}_0$ with *HP* and *JEP* there is some countable *M* such that $\mathcal{C}_0 = \mathcal{C}_0(M)$.

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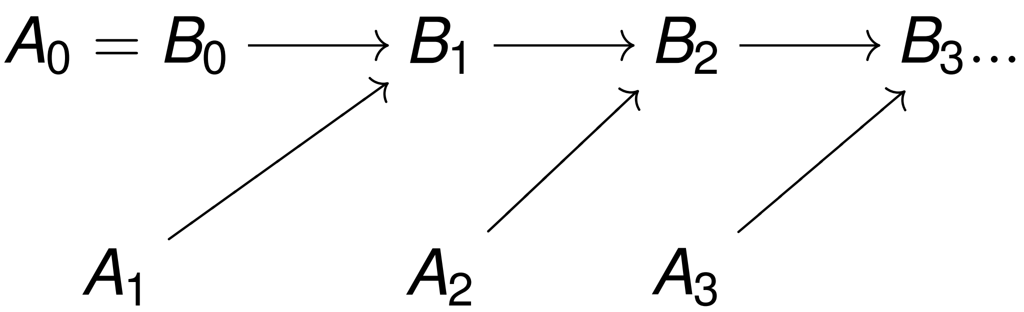
Why? Enumerate $\mathcal{C}_0 = \{A_1, A_2, ...\}$ and find B_i by *JEP* s.t.



Take $M = \bigcup_{i<\omega} B_i$. Now $\mathcal{C}_0 \subseteq \mathcal{C}_0(M)$ is clear and the converse is by *HP* and regularity of ω.

Reconstruction: first attempt

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No. Think of $(\mathbb{Z}, <)$ and $(\mathbb{Q}, <)$. They are not isomorphic, but $\mathcal{C}_0((\mathbb{Z}, <)) \cong \mathcal{C}_0((\mathbb{Q}, <))$.

How to fix?

Fixing: Homogeneity

A countable structure M is **homogeneous** if any isomorphism $A \rightarrow^{\cong} B$ in $\mathcal{C}_0(M)$ extends to an automorphism of M.

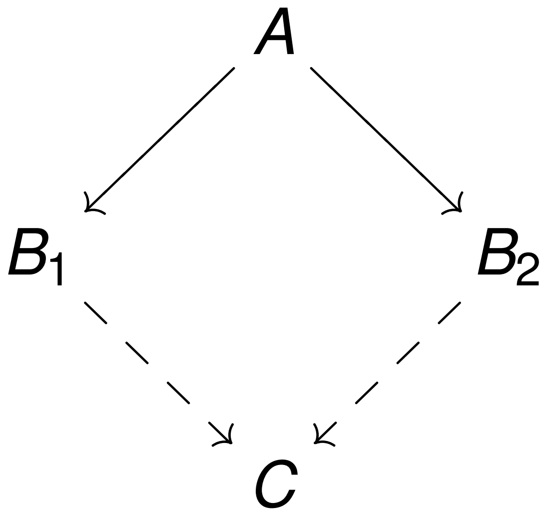
Example: $(\mathbb{Q}, <)$ is homogeneous, $(\mathbb{Z}, <)$ is not (look at the map $\{0, 1\} \rightarrow \{0, 2\}$ sending 0 to 0 and 1 to 2 – it doesn't extend to any automorphism).

Fixing: Smarter reconstruction

If M is homogeneous then $\mathcal{C}_0(M)$ also satisfies the

Amalgamation Property (AP)

For any $A, B_1, B_2 \in \mathcal{C}_0(M)$ and embeddings $A \longrightarrow B_1, A \longrightarrow B_2$ there is some $C \in \mathcal{C}_0(M)$ making the diagram commutative



(implies *JEP* if there is an initial object in $\mathcal{C}_0$, but not always)

Fraissé construction: classical version

Fix some countable language L.

Fraissé amalgamation theorem
There is a 1-to-1 correspondence:

$\{$*countable categories of finitely generated L-structures with HP, JEP and AP*$\}$ $\iff$ $\{$*countable homogeneous L-structures*$\}$

Proof Construction is like before (but a bit more careful, we have to pack B_i with all possibly amalgamable situations). Uniqueness is by back-and-forthing using homogeneity.

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Question: $\mathcal{C}_0$ - category of finite graphs.

Answer: $Fr(C_0)$ is the random graph.

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Answer: $Fr(C_0)$ is the vector space of dimension $\aleph_0$.

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Question: But if we fix some characteristic p?

Answer: $Fr(C_0)$ is the algebraically closed field of characteristic p and tr. deg. $\aleph_0$

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Answer: $Fr(C_0)$ is $(\mathbb{Q}, +)^{\aleph_0}$

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Answer: What is known as the “Hall’s universal locally finite group”. It is simple and any two isomorphic finite subgroups are conjugate.

Fraissé Amalgamation: some more examples

- C countable boolean algebras – $Fr(C_0)$ is the countable atomless boolean algebra
- partial orders – universal partial order
- C metric spaces with rational distances – completion of $Fr(C_0)$ is the universal Urishon separable metric space

Fraissé construction: Category-theoretic setting

Let C be some category and C_0 a fixed full subcategory. Suppose that

- Every ω-chain $A_1 \rightarrow A_2 \rightarrow \ldots$ in C_0 has an inverse limit in C
- Every $A \in C$ is an inverse limit of some ω-chain in C_0
- C_0 contains only countably many objects (up to isomorphism)

Then C contains a C-universal and C_0-homogeneous object if and only if C_0 satisfies *JEP* and *AP*. If such an object exists it is unique up to isomorphism and we call it $Fr(C_0)$.

So, philosophically we are finding the most general, generic object in the given category.

Hrushovski's modification

Limits tend to be wild and encode too much combinatorics, so need to be more careful about embeddings. But suppose that we want to have some nice notion of dimension on the limit. Say on elements of C_0 we have some notion of predimension and we want to be able to lift it to the limit. Then we should very carefully choose maps in our category $\mathcal{C}$, they should also preserve the dimension nicely. That was a totally handwaving and obscure way to describe the Fraissé-Hrushovski construction.

Hrushovski's modification: some examples

This is a whole industry by now:

- universal trees, hyperplanes - destroy many model-theoretic conjectures
- fusion of two algebraically closed fields or adding some strange subgroups
- A counter example to the Uryshon's conjecture
- bad fields
- more and more stuff is coming

Zilber's pseudo-exponentiation: Schanuel conjecture

Schanuel conjecture
Let $a_1, a_2, ..., a_n$ be complex numbers, linearly independent over $\mathbb{Q}$. Then $tr.deg_{\mathbb{Q}}(a_1, e^{a_1}, a_2, e^{a_2}, ..., a_n, e^{a_n}) \geq n$.

Implies all known results about trancendence of numbers, e.g.

taking $a_1 = ln2$ (clearly irrational) it would follow that $\{ln2, e^{ln2}\} = \{ln2, 2\}$ has transcendence degree at least 1, and so $ln2$ must be transcendental, a classical (and difficult) result.

Of course it is believed to be totally out of reach.

Zilber's pseudo-exponentiation

Using variant of Hrushovski's amalgamation Boris Zilber has constructed a structure $(K, +, \times, exp)$ such that

- K is an algebraically closed field of characteristic 0, *exp* is a homomorphism from $(K, +)$ to $(K, \times)$
- *exp* satisfies Schanuel conjecture
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Obvious question: are exponent and pseudo-exponent actually the same?

Moral

If you are looking for some counterexample – think of Fraissé-Hrushovski!

(it worked for me)

References (following the topics of the talk)

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Jonathan Kirby – Amalgamation constructions (check his webpage)

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