Fields and model-theoretic classification, 3

Artem Chernikov

UCLA

Model Theory conference

Stellenbosch, South Africa, Jan 12 2017

Simple theories

Definition

[Shelah] A formula $\varphi(x; y)$ has the *tree property* (TP) if there is $k < \omega$ and a tree of tuples $(a_\eta)_{\eta \in \omega^{<\omega}}$ in $\mathbb{M}$ such that:

- for all $\eta \in \omega^\omega$, $\{\varphi(x; a_{\eta|\alpha}) : \alpha < \omega\}$ is consistent,
- for all $\eta \in \omega^{<\omega}$, $\{\varphi(x; a_{\eta^\frown\langle i\rangle}) : i < \omega\}$ is k-inconsistent.

- T is *simple* if no formula has TP.
- T is *supersimple* if there is no such tree even if we allow to use a different formula $\phi_\alpha(x, y_\alpha)$ on each level $\alpha < \omega$.
- Simplicity of T admits an alternative characterization via existence of a canonical independence relation on subsets of a saturated model of T with properties generalizing those of algebraic independence (given by Shelah's forking).
- All stable theories are simple.

Pseudofinite fields

Definition

An infinite field K is *pseudofinite* if for every first-order sentence $\sigma \in \mathcal{L}_{\text{ring}}$ there is some finite field $K_0 \models \sigma$.

- Equivalently, K is elementarily equivalent to a (non-principal) ultraproduct of finite fields.
- Ax developed model theory of pseudofinite fields, in particular giving the following algebraic characterization:

Fact

[Ax, 68] A field K is pseudofinite if and only if:

1. *K is perfect,*
2. *K has a unique extension of every finite degree,*
3. *K is PAC.*

These properties are first-order axiomatizable, and completions of the theory are described by fixing the isomorphism type of the algebraic closure of the prime field.

PAC fields

- A field F is *pseudo-algebraically closed* (or *PAC*) if every absolutely irreducible variety defined over F has an F-rational point.
- A field F is *bounded* if for each $n \in \mathbb{N}$, there are only finitely many extensions of degree n.
- [Parigot] If F is PAC and not separable, then F is not NIP.
- [Beyarslan] In fact, every pseudofinite field interprets the random n-hypergraph, for all $n \in \mathbb{N}$ ($n = 2$ — Paley graphs).
- [Hrushovski], [Kim,Pillay] Every perfect bounded PAC field is supersimple.
- [Chatzidakis] A PAC field has a simple theory if and only if it is bounded.

Converse

- [Pillay, Poizat] Supersimple $\implies$ perfect and bounded.

Question [Pillay]. Is every supersimple field PAC?

- F is PAC $\iff$ the set of the F-rational points of every absolutely irreducible variety over F is Zariski-dense.
- [Geyer] Enough to show for curves over F (i.e. one-dimensional absolutely irreducible varieties over F).
- [Pillay, Scanlon, Wagner] True for curves of genus 0.
- [Pillay, Martin-Pizarro] True for (hyper-)elliptic curves with generic moduli.
- [Martin-Pizarro, Wagner] True for all elliptic curves over F with a unique extension of degree 2.
- [Kaplan, Scanlon, Wagner] An infinite field K with $\mathrm{Th}(K)$ simple has only finitely many Artin-Schreier extension (see below).

More PAC fields

- No apparent conjecture for general simple fields.
- In general, PAC fields can have wild behavior. However, there are some unbounded well-behaved PAC fields.

Definition

A field F is called *ω-free* if it has a countable elementary substructure F_0 with $\mathcal{G}(F_0) \cong \hat{\mathbb{F}}_\omega$, the free profinite group on countably many generators.

- [Chatzidakis] Not simple. However, admits a notion of independence satisfying an amalgamation theorem.
- By [C., Ramsey], this implies that if F is an ω-free PAC field, then $\mathrm{Th}(F)$ is NSOP_1.

inp-patterns and NTP_2

- T a complete theory, $\mathbb{M}$ a saturated model for T.

Definition

An inp-*pattern of depth* κ consists of $(\bar{a}_\alpha, \varphi_\alpha(x, y_\alpha), k_\alpha)_{\alpha\in\kappa}$ with $\bar{a}_\alpha = (a_{\alpha,i})_{i\in\omega}$ from $\mathbb{M}$ and $k_\alpha \in \omega$ such that:

- $\{\varphi_\alpha(x, a_{\alpha,i})\}_{i\in\omega}$ is k_α-inconsistent for every $\alpha \in \kappa$,
- $\left\{\varphi_\alpha(x, a_{\alpha,f(\alpha)})\right\}_{\alpha\in\kappa}$ is consistent for every $f : \kappa \to \omega$.

- The *burden* of T is the supremum of the depths of inp-patterns with x a singleton, either a cardinal or ∞.
- T is NTP_2 if burden of T is $< \infty$. Equivalently, if there is no inp-pattern of infinite depth with the same formula and k on each row.
- T is *strong* if there is no infinite inp-pattern.
- T is inp-*minimal* if there is no inp-pattern of depth 2, with $|x| = 1$.
- Retroactively, T is *dp-minimal* if it is NIP and inp-minimal.

inp-patterns and NTP_2

- T is simple or NIP $\implies$ T is NTP_2 (exercise).
- [C., Kaplan], [Ben Yaacov, C.], etc. There is a theory of forking in NTP_2 theories (generalizing the simple case).
- There are many new algebraic examples in this class!

Examples of NTP_2 fields: ultraproducts of p-adics

- We saw that for every prime p, the field $\mathbb{Q}_p$ is NIP.
- However, consider the field $\mathcal{K} = \prod_{p \text{ prime}} \mathbb{Q}_p / \mathcal{U}$ (where $\mathcal{U}$ is a non-principal ultrafilter on the set of prime numbers) — a central object in the applications of model theory, after [Ax-Kochen], [Denef-Loeser], ...
- The theory of $\mathcal{K}$ is not simple: because the value group is linearly ordered.
- The theory of $\mathcal{K}$ is not NIP: the residue field is pseudofinite.
- Both already in the pure ring language, as the valuation ring is definable uniformly in p [e.g. Ax].

Ax-Kochen principle for NTP_2

- Delon's transfer theorem for NIP has an analog for NTP_2 as well.

Theorem

[C.] Let $\mathcal{K} = (K, \Gamma, k, v, \mathrm{ac})$ be a henselian valued field of equicharacteristic 0, in the Denef-Pas language. Assume that k is NTP_2. Then $\mathcal{K}$ is NTP_2.

- Being strong is preserved as well.

Corollary

$\mathcal{K} = \prod_{p \text{ prime}} \mathbb{Q}_p / \mathcal{U}$ is NTP_2 because the residue field is pseudofinite, hence simple, hence NTP_2.

- More recently, [C., Simon]. $\mathcal{K}$ is inp-minimal in $\mathcal{L}_{\mathrm{ring}}$ (but not in the language with ac, of course).

Valued difference fields, 1

- $(K, \Gamma, k, v, \sigma)$ is a *valued difference field* if $(K, \Gamma, k, v, \mathrm{ac})$ is a valued field and σ is a field automorphism preserving the valuation ring.
- Note: σ induces natural automorphisms on k and on Γ.
- Because of the order on the value group, by [Kikyo,Shelah] there is no model companion of the theory of valued difference fields.
- The automorphism σ is *contractive* if for all $x \in K$ with $v(x) > 0$ we have $v(\sigma(x)) > nv(x)$ for all $n \in \omega$.
- **Example**: Let $(K_p, \Gamma, k, v, \sigma)$ be an algebraically closed valued field of char p with σ interpreted as the Frobenius automorphism. Then $\prod_{p \text{ prime}} K_p/\mathcal{U}$ is a contractive valued difference field.

Valued difference fields, 2

[Hrushovski], [Durhan] Ax-Kochen-Ershov principle for σ-henselian contractive valued difference fields $(K, \Gamma, k, v, \sigma, \mathrm{ac})$:

- Elimination of the field quantifier.
- $(K, \Gamma, k, v, \sigma) \equiv (K', \Gamma', k', v, \sigma)$ iff $(k, \sigma) \equiv (k', \sigma)$ and $(\Gamma, <, \sigma) \equiv (\Gamma', <, \sigma)$;
- There is a model companion VFA_0 and it is axiomatized by requiring that $(k, \sigma) \models \mathrm{ACFA}_0$ and that $(\Gamma, +, <, \sigma)$ is a divisible ordered abelian group with an ω-increasing automorphism.
- Nonstandard Frobenius is a model of VFA_0.
- The reduct to the field language is a model of ACFA_0, hence simple but not NIP. On the other hand this theory is not simple as the valuation group is definable.

Valued difference fields and NTP_2

Theorem

[C., Hils] Let $\bar{K} = (K, \Gamma, k, v, ac, \sigma)$ be a σ-Henselian contractive valued difference field of equicharacteristic 0. Assume that both (K, σ) and (Γ, σ), with the induced automorphisms, are NTP_2. Then $\bar{K}$ is NTP_2.

Corollary

VFA_0 is NTP_2 (as $ACFA_0$ is simple and $(\Gamma, +, <, \sigma)$ is NIP).

- The argument also covers the case of σ-henselian valued difference fields with a value-preserving automorphism of [Belair, Macintyre, Scanlon] and the multiplicative generalizations of Kushik.
- Open problem: is VFA_0 strong?

PRC fields, 1

- F is PAC $\iff$ M is existentially closed (in the language of rings) in each regular field extension of F.

Definition

[Basarab, Prestel] A field F is *Pseudo Real Closed* (or PRC) if F is existentially closed (in the ring language) in each regular field extension F' to which all orderings of F extend.

- Equivalently, for every absolutely irreducible variety V defined over F, if V has a simple rational point in every real closure of F, then V has an F-rational point.
- E.g. PAC (has no orderings) and real closed fields are PRC (no proper real closures).
- The class of PRC fields is elementary.
- Were studied by Prestel, Jarden, Basarab, McKenna, van den Dries and others.

PRC fields, 2

- If K is a bounded field, then it has only finitely many orders (bounded by the number of extensions of degree 2).
- [Chatzidakis] If a PAC field is not bounded, then it has TP_2. Easily generalizes to PRC.
- **Conjecture** [C., Kaplan, Simon]. A PRC field is NTP_2 if and only if it is bounded (and the same for PpC fields).

Fact

[Montenegro, 2015] A PRC field K is bounded if and only if $\mathrm{Th}(K)$ *is* NTP_2*.*
Moreover, the burden of K is equal to the number of the orderings.

PpC fields

- A valuation (F, v) is *p-adic* if the residue field is $\mathbb{F}_p$ and $v(p)$ is the smallest positive element of the value group.

Definition

[Grob, Jarden and Haran] F is pseudo p-adically closed (PpC) if F is existentially closed (in $\mathcal{L}_{\text{ring}}$) in each regular extension F' such that all the p-adic valuations of M can be extended by p-adic valuations on F'.

Fact

[Montenegro, 2015] All bounded PpC fields are NTP_2.

- The converse is still open.

NTP_2 fields have finitely many Artin-Schreier extensions

- What do we know about general NTP_2 fields?
- Generalizing the simple case, we have:

Theorem

[C., Kaplan, Simon] Let K be an infinite NTP_2 field. Then it has only finitely many Artin-Schreier extensions.

Corollary

$\mathbb{F}_p((t))$ has TP_2.

Ingredients of the proof

1. The proof generalizes the arguments in [Kaplan-Scanlon-Wagner] for the NIP case, using a new chain condition for NTP_2 groups.
2. Let G be NTP_2 and $\{\varphi(x,a) : a \in C\}$ be a family of normal subgroups of G. Then there is some $k \in \omega$ (depending only on φ) such that for every finite $C' \subseteq C$ there is some $C_0 \subseteq C'$ with $|C_0| \leq k$ and such that

$$\left[\bigcap_{a \in C_0} \varphi(x,a) : \bigcap_{a \in C'} \varphi(x,a)\right] < \infty.$$

3. Open problem: does it hold without the normality assumption?

Definable envelopes of groups in NTP_2

- A group G is finite-by-abelian if there exists a finite normal subgroup F of G such that G/F is abelian.
- If $H, K \leq G$, H is *almost contained in* K if $[H : H \cap K]$ is finite.
- Generalizing the results of Poizat, Shelah, de Aldama, Milliet from stable, simple and NIP cases:

Fact

[Hempel, Onshuus] Let G be a group definable in an NTP_2 *theory, H a subgroup of G (not necessarily definable!) and*

- If H is abelian (nilpotent of class n), then there exists a *definable* finite-by-abelian (resp. nilpotent of class $\leq 2n$) subgroup H' of G which contains (resp. almost contains) H. If H was normal, can choose H' normal as well.
- If H is a normal solvable subgroup of class n, there exists a definable normal solvable subgroup H' of G of class at most $2n$ which almost contains H.