Fields and model-theoretic classification, 2

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NIP

Definition

Let T be a complete first-order theory in a language $\mathcal{L}$.

1. A (partitioned) formula $\phi(x, y)$ is *NIP* (No Independence Property) if there are no $\mathcal{M} \models T$ and $(a_i)_{i \in \mathbb{N}}$ from $M^{|x|}$ and $(b_J)_{J \subseteq \mathbb{N}}$ such that

$$\mathcal{M} \models \phi(a_i, b_J) \iff i \in J.$$

2. T is NIP if it implies that all formulas are NIP.
3. $\mathcal{M}$ is NIP if $\mathrm{Th}(\mathcal{M})$ is NIP.

- The class of NIP theories was introduced by Shelah, later noticed by Laskowski that $\phi(x, y)$ is NIP $\iff$ $\{\phi(M, b) : b \in M^{|y|}\}$ has finite Vapnik-Chervonenkis dimension from statistical learning theory.
- Attracted a lot of attention in model theory (new important algebraic examples + generalizing methods of stability).

Examples of NIP theories

- T stable $\implies$ T is NIP.
- [Shelah] T is NIP if all formulas $\phi(x, y)$ with $|x| = 1$ are NIP.
- Using it (and that Boolean combinations of NIP formulas are NIP), easy to see that every *o*-minimal theory is NIP*.
  In particular, $(\mathbb{R}, +, \cdot, 0, 1)$.
- $(\mathbb{Q}_p, +, \cdot, 0, 1)$ eliminates quantifiers in the language expanded by $v(x) \leq v(y)$ and $P_n(x) \iff \exists y (x = y^n)$ for all $n \geq 2$ [Macintyre]. Using it, not hard to check NIP.

More examples: Delon's theorem, etc.

Fact

[Delon] + [Gurevich-Schmitt] Let (K, v) *be a henselian valued field of residue characteristic* $\mathrm{char}\,(k) = 0$. *Then* (K, v) *is NIP* $\iff$ k *is NIP (as a pure field).*

- Can also work in the Denef-Pas language or in the RV language.
- Various versions in positive characteristic: Belair, Jahnke-Simon.

What do we know about NIP fields

- Are these all examples?
- **Conjecture.** [Shelah, and others] Let K be an NIP field. Then K is either separably closed, or real closed, or admits a non-trivial henselian valuation.
- [Johnson] In the dp-minimal case, yes (see later).
- In general, what do we know about NIP fields (and groups)?

Definable families of subgroups Baldwin-Saxl

- Let G be a group definable in an NIP structure $\mathcal{M}$.
- By a *uniformly definable family* of subgroups of G we mean a family of subgroups $(H_i : i \in I)$ of G such that for some $\phi(x, y)$ we have $H_i = \phi(M, a_i)$ for some parameter a_i, for all $i \in I$.

Baldwin-Saxl, 1

Fact

[Baldwin-Saxl] Let G be an NIP *group. For every formula $\phi(x, y)$ there is some number $m = m(\phi) \in \omega$ such that if I is finite and $(H_i : i \in I)$ is a uniformly definable family of subgroups of G of the form $H_i = \phi(\mathbb{M}, a_i)$ for some parameters a_i, then $\bigcap_{i \in I} H_i = \bigcap_{i \in I_0} H_i$ for some $I_0 \subseteq I$ with $|I_0| \leq m$.*

Proof.

Otherwise for each $m \in \omega$ there are some subgroups $(H_i : i \leq m)$ such that $H_i = \phi(\mathbb{M}, a_i)$ and $\bigcap_{i \leq m} H_i \subsetneq \bigcap_{i \leq m, i \neq j} H_i$ for every $j \leq m$. Let b_j be an element from the set on the right hand side and not in the set on the left hand side. Now, if $I \subseteq \{0, 1, \ldots, m\}$ is arbitrary, define $b_I := \prod_{j \in I} b_j$. It follows that $\models \phi(b_I, a_i) \iff i \notin I$. This implies that $\phi(x, y)$ is not NIP. $\square$

Connected components and generics

- As in the ω-stable case, implies existence of connected components: G^0, G^{00}, G^∞ (however, now G^0 is only type-definable).
- Study of groups in NIP brings to the picture connections to topological dynamics, measure theory, etc (G/G^{00} is a compact topological group explaining a lot about G itself).
- [Hrushovski, Pillay], [C., Simon] *Definably amenable* NIP groups admit a satisfactory theory of generics (generalizing stable and *o*-minimal cases).

Artin-Schreier extensions

- Let k be a field, $\mathrm{char}(k) = p$. Let ρ be the polynomial $X^p - X$.

Fact

[Artin-Schreier]

1. *Given $a \in k$, either the polynomial $\rho - a$ has a root in k, in which case all its roots are in k, or it is irreducible.*
   In the latter case, if α is a root then $k(\alpha)$ is cyclic of degree p over k.
2. *Conversely, let K be a cyclic extension of k of degree p. Then there exists $\alpha \in K$ such that $K = k(\alpha)$ and for some $a \in k$, $\rho(\alpha) = a$.*

- *Such extensions are called Artin-Schreier extensions.*

NIP fields are Artin-Schreier closed, 1

Fact

[Kaplan-Scanlon-Wagner, 2010] Let K be an infinite NIP field of characteristic $p > 0$. Then K is Artin-Schreier closed (i.e. no proper A-S extensions, that is ρ is onto).

- [Hempel, 2015] generalized this to *n-dependent* fields.
- We will sketch the proof in the NIP case.

Corollary

If L/K is a Galois extension, then p does not divide $[L : K]$.

Corollary

K contains $\mathbb{F}_p^{\mathrm{alg}}$.

NIP fields are Artin-Schreier closed, 2

1. Let F be an algebraically closed field containing K.
2. For $n \in \mathbb{N}$ and $\bar{b} \in F^{n+1}$, define

$$G_{\bar{b}} := \left\{ (t, x_1, \ldots, x_n) : t = b_i \left(x_i^p - x_i \right) \text{ for } 1 \leq i \leq n \right\}.$$

3. $G_{\bar{b}}$ is an algebraic subgroup of $(F, +)^{n+1}$.
4. If $\bar{b} \in K$, then by Baldwin-Saxl, for some $n_0 \in \mathbb{N}$, for every finite tuple $\bar{b}$, there is a sub-n_0-tuple $\bar{b}'$ such that the projection $\pi : G_{\bar{b}}(K) \to G_{\bar{b}'}(K)$ is onto.
   (Consider the family of subgroups of $(K, +)$ of the form $\{t : \exists x \, t = a(x^p - x)\}$ for $a \in K$.)

NIP fields are Artin-Schreier closed, 3

Claim 1. Let F be an algebraically closed field. Suppose $\bar{b} \in F^{\times}$ is algebraically independent, then $G_{\bar{b}}$ is a connected group.

Claim 2. Let F be an algebraically closed field of characteristic p, and let $f : F \to F$ be an additive polynomial (i.e. $f(x+y) = f(x) + f(y)$). Then f is of the form $\sum a_i x^{p^i}$. Moreover, if $\ker(f) = \mathbb{F}_p$ then $f = (a(x^p - x))^{p^n}$ for some $n < \omega, a \in F^{\times}$.

Fact. Let k be a perfect field, and G a closed 1-dimensional connected algebraic subgroup of $\left(k^{\mathrm{alg}}, +\right)^n$ defined over k, for some $n < \omega$. Then G is isomorphic over k to $\left(k^{\mathrm{alg}}, +\right)$.

NIP fields are Artin-Schreier closed, 4

- We may assume that K is $\aleph_0$-saturated.
- Let $k = \bigcap_{n \in \omega} K^{p^n}$, k is an infinite perfect field.
- Choose an algebraically independent tuple $\bar{b} \in k^{n_0+1}$.
- By Baldwin-Saxl, there is some sub-n_0-tuple $\bar{b}'$ such that the projection $\pi : G_{\bar{b}}(K) \to G_{\bar{b}'}(K)$ is onto.
- By the first claim, both $G_{\bar{b}}$ and $G_{\bar{b}'}$ are connected. And their dimension is 1.

NIP fields are Artin-Schreier closed, 5

- By the Fact, both these groups are isomorphic over k to $\left(K^{\text{alg}}, +\right)$.
- So we have some $\nu \in k[x]$ such that

$$\begin{array}{ccc} G_{\bar{b}}(K^{alg}) & \xrightarrow{\pi} & G_{\bar{b}'}(K^{alg}) \\ \downarrow & & \downarrow \\ (K^{alg},+) & \xrightarrow{\nu} & (K^{alg},+) \end{array}$$

commutes.

- As the sides are isomorphisms defined over $k \subseteq K$, we can restrict them to K. As $\pi \restriction G_{\bar{b}}(k)$ is onto $G_{\bar{b}'}(K)$, then so is $\nu \restriction K$.
- $|\ker(\nu)| = p = |\ker(\pi)|$ (even when restricted to k).

NIP fields are Artin-Schreier closed, 6

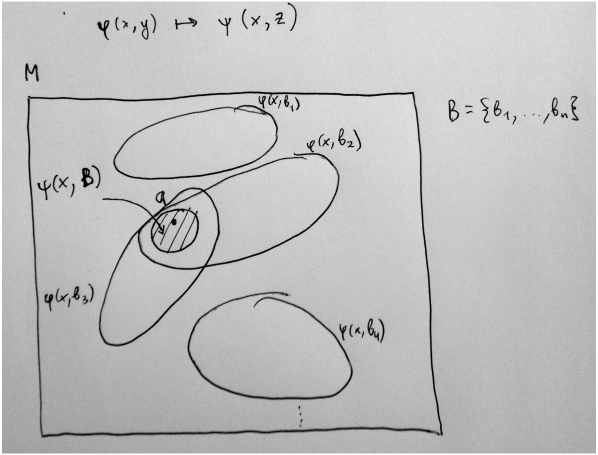
- Suppose that $0 \neq c \in \ker(\nu) \subseteq k$. Let $\nu' := \nu \circ m_c$, where $m_c(x) = c \cdot x$.
- ν' is an additive polynomial over K whose kernel is $\mathbb{F}_p$. So WLOG $\ker(\nu) = \mathbb{F}_p$.
- By Claim 2, ν is of the form $a \cdot (x^p - x)^{p^n}$ for $a \in K^\times$.
- But ν is onto, hence so is ρ (given $y \in K$, there is some $x \in K$ such that $a \cdot (x^p - x)^{p^n} = a \cdot y^{p^n}$).

Distal structures, 1

- The class of *distal theories* was introduced by [Simon, 2011] in order to capture the class of NIP structures without any infinite stable "part".
- The original definition is in terms of a certain property of indiscernible sequences.
- [C., Simon, 2012] give a combinatorial characterization of distality:

Distal structures, 2

- **Theorem/Definition** An NIP structure M is *distal* if and only if for every definable family $\{\phi(x, b) : b \in M^d\}$ of subsets of M there is a definable family $\{\psi(x, c) : c \in M^{kd}\}$ such that for every $a \in M$ and every finite set $B \subset M^d$ there is some $c \in B^k$ such that $a \in \psi(x, c)$ and for every $a' \in \psi(x, c)$ we have $a' \in \phi(x, b) \Leftrightarrow a \in \phi(x, b)$, for all $b \in B$.



Examples of distal structures

- Distality can be thought of as a combinatorial abstraction of a cell decomposition.
- All (weakly) *o*-minimal structures, e.g. $M = (\mathbb{R}, +, \times, e^x)$.
- Presburger arithmetic.
- Any *p*-minimal theory with Skolem functions is distal. E.g. $(\mathbb{Q}_p, +, \times)$ for each prime p is distal (e.g. due to the *p*-adic cell decomposition of Denef).
- [Aschenbrenner, C.] The (valued differential) field of transseries.
  Also, an analog of Delon's theorem holds for distality.

Example: o-minimal implies distal

- Let $\mathcal{M}$ be *o*-minimal and $\phi(x, \bar{y})$ given.
- For any $\bar{b} \in M^{|\bar{y}|}$, $\phi\left(x, \bar{b}\right)$ is a finite union of intervals whose endpoints are of the form $f_i\left(\bar{b}\right)$ for some definable functions $f_0(\bar{y}), \ldots, f_k(\bar{y})$.
- Given a finite set $B \subseteq M^{|\bar{y}|}$, the set of points $\left\{f_i\left(\bar{b}\right) : i < k, \bar{b} \in B\right\}$ divides M into finitely many intervals, and any two points in the same interval have the same ϕ-type over B.
- Thus, for any $a \in M$, either $a = f_i\left(\bar{b}\right)$ for some $i < k$ and $\bar{b} \in B$, or $f_i\left(\bar{b}\right) < x < f_j\left(\bar{b}'\right) \vdash \mathrm{tp}_\phi(a/B)$ for some $i, j < k$ and $\bar{b}, \bar{b}' \in B$.

Strong Erdős-Hajnal property in distal structures, 1

Fact

[C., Starchenko, 2015]

1. *Let $\mathcal{M}$ be a distal structure. For every definable relation $R \subseteq M^{d_1} \times M^{d_2}$ there is some real $\varepsilon > 0$ such that: for every finite $A \subseteq M^{d_1}, B \subseteq M^{d_2}$ there are some $A' \subseteq A, B' \subseteq B$ such that $|A'| \geq \varepsilon |A|, |B'| \geq \varepsilon |B|$ and (A', B') is R-homogeneous.*
   Moreover, $A' = A \cap S_1$ and $B' = B \cap S_2$, where S_1, S_2 are definable by an instance of a certain formula depending just on the formula defining R (and not on its parameters).
2. *Conversely, if all definable relations in $\mathcal{M}$ satisfy this property, then $\mathcal{M}$ is distal.*

Strong Erdős-Hajnal property in distal structures, 2

- Part (1) was known in the following special cases:
- [Alon, Pach, Pinchasi, Radoičić, Sharir, 1995]
  $\mathcal{M} = (\mathbb{R}, +, \times, 0, 1)$.
- [Basu, 2007] Topologically closed graphs in *o*-minimal expansions of real closed fields.

Strong EH fails in ACF_p

- Without requiring definability of the homogeneous sets, strong EH holds in algebraically closed fields of char 0 — as $(\mathbb{C}, \times, +)$ is interpreted in $\left(\mathbb{R}^2, \times, +\right)$.
- For a finite field $\mathbb{F}_q$, let P_q be the set of all points in $\mathbb{F}_q^2$ and let L_q be the set of all lines in $\mathbb{F}_q^2$.
- Let $I \subseteq P_q \times L_q$ be the incidence relation. Using the fact that the bound $|I(P_q, L_q)| \leq |L_q| |P_q|^{\frac{1}{2}} + |P_q|$ is known to be optimal in finite fields, one can check:
- **Claim**. For any fixed $\delta > 0$, for all large enough q if $L_0 \subseteq L_q$ and $P_0 \subseteq P_q$ with $|P_0| \geq \delta q^2$ and $|L_0| \geq \delta q^2$ then $I(P_0, L_0) \neq \emptyset$.
- As every finite field of char p can be embedded into $\mathbb{F}_p^{\mathrm{alg}}$, it follows that strong EH fails in $\mathbb{F}_p^{\mathrm{alg}}$ (even without requiring definability of the homogeneous pieces) for I the incidence relation.

Infinite distal fields have characteristic 0

- As by Kaplan, Scanlon, Wagner, every NIP field of positive characteristic p contains $\mathbb{F}_p^{\mathrm{alg}}$, we have:

Corollary

[C., Starchenko] Every infinite field interpretable in a distal structure has characteristic 0*.*

dp-minimal fields, 1

Definition

A theory T is *dp-minimal* if it is NIP of "rank 1" (will define precisely later).

- Examples of dp-minimal theories: strongly minimal, (weakly) o-minimal, C-minimal, finite extensions of $\mathbb{Q}_p$, Henselian valued fields of char $(0, 0)$ with dp-minimal residue field and value group.

Fact

[Johnson, 2015] Let K be a dp-minimal field. Then K is either algebraically closed, or real closed, or admits a non-trivial henselian valuation (+ a more explicit characterization of dp-minimal fields).

dp-minimal fields, 2

- Key ideas:
  - Given an infinite dp-minimal field K which is not strongly minimal, then $\{X - X : X \subseteq K \text{ definable and infinite}\}$ is a neighborhood basis of 0 for a Hausdorff non-discrete definable field topology such that if $0 \notin \overline{X}$, then $0 \notin \overline{X \cdot X}$ for $X \subseteq K$ (so called V-topology).
  - By [Kowalsky, Dürbaum], such a topology must arise from a non-trivial valuation or an absolute value on K (need not be definable or unique).
- Let $\mathcal{O}$ be the intersection of all $\emptyset$-definable valuation rings on K. Then, in particular relying on [Jahnke, Koenigsmann, 2015] work on defining henselian valuations, $\mathcal{O}$ is henselian, induces the canonical topology on K and the residue field is finite, algebraically closed, or real closed.