Fields with NTP_2

Artem Chernikov

Hebrew University of Jerusalem

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Outline

Shelah's classification theory and NTP_2

Examples of fields with NTP_2

Implications of NTP_2 for properties of definable groups and fields

Quantitative refinements of NTP_2 — burden, strongness, inp-minimality

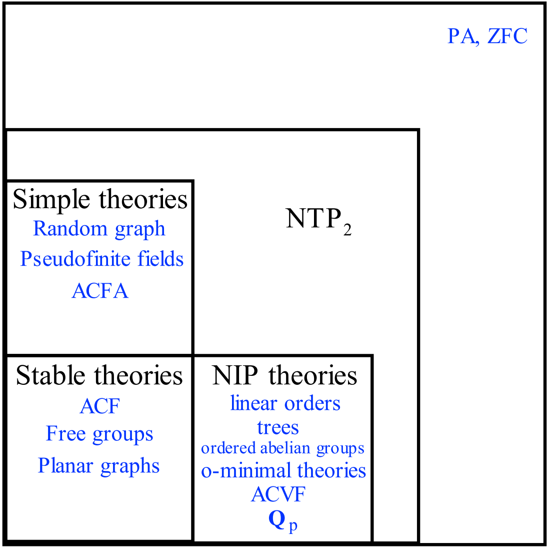
Some history

- We consider complete first-order theories in a countable language, $\mathbb{M}$ denotes a monster model.
- Shelah's philosophy of dividing lines — classify complete first-order theories by their ability to encode certain combinatorial configurations. He identified several very concrete configurations (e.g. linear order in the case of stability) such that:
  - when the theory cannot encode them, the category of definable sets and types admits a coherent theory (forking, ranks, weight, analyzability, etc leading to a classification of models);
  - when it can, one can prove a non-structure result (many models in the case of stability).
- In algebraic situations such as groups or fields, these model-theoretic properties turn out to be closely related to algebraic properties of the structure.
- Later work of Zilber, Hrushovski and others on geometric stability theory produced deep aplications to purely algebraic questions.

Some history

- Unfortunately, most structures studied in mathematics are not stable.
- Simple theories: developed by Shelah, Hrushovski, Kim, Pillay, Chatzidakis, Wagner and others. Applications in algebraic dynamics, etc.
- Various minimality settings: o-minimality, c-minimality, p-minimality, etc — concentrated on definable sets rather than types, not quite in the spirit of stability theory.
- Common context to treat these settings — NIP: Pillay's conjecture on groups in o-minimal theories, work of Haskell, Hrushovski and Macpherson on algebraically closed valued fields and stable domination.

Shelah's classification theory and generalizations of stability



NTP_2

Definition

[Shelah]

1. A formula $\phi(x, y)$, where x and y are tuples of variables, has TP_2 (*Tree Property of the 2nd kind*) if there is an array $(a_{i,j})_{i,j\in\omega}$ of tuples from $\mathbb{M}$ and $k \in \omega$ such that:
   - $\{\phi(x, a_{i,j})\}_{j\in\omega}$ is k-inconsistent for every $i \in \omega$.
   - $\left\{\phi\left(x, a_{i,f(i)}\right)\right\}_{i\in\omega}$ is consistent for every $f : \omega \to \omega$.
2. A theory is NTP_2 if it implies that no formula has TP_2.

Fact

[Ch.] Enough to check formulas with $|x| = 1$.

Fact

Every simple or NIP *theory is* NTP_2.

NTP_2

- In [Ch., Kaplan] and later [Ben Yaacov, Ch.] a reasonable theory of forking over extension bases in NTP_2 theories was developed:
  - encorporates the theory of forking in simple theories due to Kim, Pillay, Hrushovski and others as a special case;
  - provides answers to some questions of Pillay and Adler around forking and dividing in the case of NIP.
- Guiding principle (rather naive) — NTP_2 is a combination of simple and NIP (e.g. densely ordered random graph, the model companion of the theory of ordered graphs, is neither simple nor NIP; but it is NTP_2).

Examples of NTP_2 fields: ultraproducts of p-adics

- For every prime p, the valued field $(\mathbb{Q}_p, +, \times, 0, 1)$ is NIP.
- However, consider the valued field $\mathcal{K} = \prod_{p \text{ prime}} \mathbb{Q}_p / \mathcal{U}$ (where $\mathcal{U}$ is a non-principal ultrafilter on the set of prime numbers) — a central object in the model theoretic applications to valued fields after the work of Ax and Kochen.
- The theory of $\mathcal{K}$ is not simple: because the value group is linearly ordered.
- The theory of $\mathcal{K}$ is not NIP: the residue field is pseudofinite, thus has the independence property by a result of Duret.
- Both even in the pure ring language: as the valuation ring is definable uniformly in p (Ax).
- Canonical models: Hahn fields of the form $k\left(\left(t^{\mathbb{Z}}\right)\right)$, where k is a pseudofinite field.

Ax-Kochen principle for NTP_2

Fact

[Delon + Gurevich, Schmitt] Let $\mathcal{K} = (K, \Gamma, k, v, \mathrm{ac})$ *be a henselian valued field of equicharacteristic* 0*, in the Denef-Pas language. Assume that* k *is NIP. Then* $\mathcal{K}$ *is NIP.*

Theorem

[Ch.] Let $\mathcal{K} = (K, \Gamma, k, v, \mathrm{ac})$ *be a henselian valued field of equicharacteristic* 0*, in the Denef-Pas language. Assume that* k *is* NTP_2*. Then* $\mathcal{K}$ *is* NTP_2*.*

Corollary

$\mathcal{K} = \prod_{p \text{ prime}} \mathbb{Q}_p / \mathcal{U}$ *is* NTP_2 *because the residue field is pseudofinite, so simple, so* NTP_2*.*

Problem: Show an analogue for positive characteristic (Belair for NIP).

Valued difference fields

- $(K, \Gamma, k, v, \sigma)$ is a *valued difference field* if $(K, \Gamma, k, v, \mathrm{ac})$ is a valued field and σ is a field automorphism preserving the valuation ring.
- Note that σ induces natural automorphisms on k and on Γ.
- Because of the order on the value group, it follows by [Kikyo,Shelah] the there is no model companion of the theory of valued difference fields.
- The automorphism σ is *contractive* if for all $x \in K$ with $v(x) > 0$ we have $v(\sigma(x)) > nv(x)$ for all $n \in \omega$.
- **Example**: Let $(F_p, \Gamma, k, v, \sigma)$ be an algebraically closed valued field of char p with σ interpreted as the Frobenius automorphism. Then $\prod_{p \text{ prime}} F_p/\mathcal{U}$ is a contractive valued difference field.

Valued difference fields

[Hrushovski], [Durhan] Ax-Kochen principle for σ-henselian contractive valued difference fields $(K, \Gamma, k, v, \sigma, \mathrm{ac})$:

- Elimination of the field quantifier;
- $(K, \Gamma, k, v, \sigma) \equiv (K', \Gamma', k', v, \sigma)$ iff $(k, \sigma) \equiv (k', \sigma)$ and $(\Gamma, <, \sigma) \equiv (\Gamma', <, \sigma)$;
- There is a model companion VFA_0 and it is axiomatized by requiring that $(k, \sigma) \models \mathrm{ACFA}_0$ and that $(\Gamma, +, <, \sigma)$ is a divisible ordered abelian group with an ω-increasing automorphism.
- Nonstandard Frobenius is a model of VFA_0.
- The reduct to the field language is a model of ACFA_0, hence simple but not NIP. On the other hand this theory is not simple as the valuation group is definable.

Valued difference fields and NTP_2

Theorem

[Ch.-Hils] Let $\bar{K} = (K, \Gamma, k, v, ac, \sigma)$ be a σ-Henselian contractive valued difference field of equicharacteristic 0. Assume that both (K, σ) and (Γ, σ), with the induced automorphisms, are NTP_2. Then $\bar{K}$ is NTP_2.

Corollary

VFA_0 is NTP_2 (as $ACFA_0$ is simple and $(\Gamma, +, <, \sigma)$ is NIP).

- **Conjecture**: One can ommit the requirement on the value group.
- Besides, our argument also covers the case of σ-henselian valued difference fields with a value-preserving automorphism of [Belair, Macintyre, Scanlon] and the multiplicative generalizations of Kushik.

Some conjectural examples

- A field is pseudo algebraically closed (PAC) if every absolutely irreducible variety defined over it has a point in it.
- It is well-known that the theory of a PAC field is simple if and only if it is bounded (i.e. for any integer n it has only finitely many Galois extensions of degree n). Moreover, if a PAC field is unbounded, then it has TP_2 [Chatzidakis].
- On the other hand, the following fields were studied extensively:
  1. Pseudo real closed (or PRC) fields: a field F is PRC if every absolutely irreducible variety defined over F that has a rational point in every real closure of F, has an F-rational point.
  2. Pseudo p-adically closed (or PpC) fields: a field F is PpC if every absolutely irreducible variety defined over F that has a rational point in every p-adic closure of F, has an F-rational point.
- **Conjecture**: A PRC field is NTP_2 if and only if it is bounded. Similarly, a PpC field is NTP_2 if and only if it is bounded.

Algebraic properties from tameness assumptions

- [Macintyre] Every ω-stable field is algebraically closed.
- [Cherlin-Shelah] Every superstable field is algebraically closed.
- **Conjecture**: Every stable field is separably closed.
- Many further results: every o-minimal field is real-closed, every C-minimal valued field is algebraically closed, etc...

Algebraic properties beyond stability

- Recall that given a field K of characteristic $p > 0$, an extension L/K is *Artin-Schreier* if $L = K(\alpha)$ for some $\alpha \in L \setminus K$ such that $\alpha^p - \alpha \in K$.
- [Kaplan, Scanlon, Wagner]:
  1. Let K be an NIP field. Then it is Artin-Schreier closed.
  2. Let K be a (type-definable) simple field. Then it has only finitely many Artin-Schreier extensions.
- Remember our guiding principle: $\mathrm{NTP}_2 \sim \mathrm{NIP} + \text{simple}$.

NTP_2 fields have finitely many Artin-Schreier extensions

Theorem

[Ch., Kaplan, Simon] Let K be a field definable in an NTP_2 *structure. Then it has only finitely many Artin-Schreier extensions.*

- Type-definable case is open even for NIP theories.

Ingredients of the proof

1. [Kaplan-Scanlon-Wagner] For a perfect field K of characteristic p, given a tuple of algebraically independent elements $\bar{a} = (a_1, \ldots, a_n)$ from K and some large algebraically closed extension $\mathbf{K}$, the group $G_{\bar{a}} = \left\{(t, x_1, \ldots, x_n) \in \mathbf{K}^{n+1} : t = a_i \left(x_i^p - x_i\right) \text{ for } 1 \leq i \leq n\right\}$ is algebraically isomorphic over K to $(\mathbf{K}, +)$.
2. Chain condition for uniformly definable normal subgroups: Let G be NTP$_2$ and $\{\varphi(x, a) : a \in C\}$ be a family of normal subgroups of G. Then there is some $k \in \omega$ (depending only on φ) such that for every finite $C' \subseteq C$ there is some $C_0 \subseteq C'$ with $|C_0| \leq k$ and such that

$$\left[\bigcap_{a \in C_0} \varphi(x, a) : \bigcap_{a \in C'} \varphi(x, a)\right] < \infty.$$

3. Combine.

Quantitative measure of NTP_2: burden

Definition

1. An inp-*pattern of depth* κ consists of $(\bar{a}_\alpha, \varphi_\alpha(x, y_\alpha), k_\alpha)_{\alpha\in\kappa}$ with $\bar{a}_\alpha = (a_{\alpha,i})_{i\in\omega}$ and $k_\alpha \in \omega$ such that:
   - $\{\varphi_\alpha(x, a_{\alpha,i})\}_{i\in\omega}$ is k_α-inconsistent for every $\alpha \in \kappa$,
   - $\left\{\varphi_\alpha(x, a_{\alpha,f(\alpha)})\right\}_{\alpha\in\kappa}$ is consistent for every $f : \kappa \to \omega$.
2. The *burden* of T is the supremum of the depths of inp-patterns with x a singleton, computed in Card^*.

Quantitative measure of NTP_2: burden

Possible values of the burden of a theory in a countable language:

1. $n \in \omega \setminus \{0\}$ — there is no inp-pattern of depth $\geq n$;
2. $\aleph_0^-$ — there are patterns of arbitrary finite depth, but not of infinite depth. Theories with this burden are called *strong*;
3. $\aleph_0$ — there is an inp-pattern of infinite depth, but not of arbitrary large depth. This means that a theory is NTP_2, but not strong;
4. ∞ — there are inp-patterns of depth κ for any cardinal κ. This is equivalent to TP_2 by compactness.

Burden of pseudo-local valued fields

Definition

Theories of burden 1 are called inp-minimal.

Theorem

[Ch., finer version] Let $\mathcal{K} = (K, \Gamma, k, v, \mathrm{ac})$ be a henselian valued field of equicharacteristic 0*, in the Denef-Pas language. Assume that k and Γ are strong (of finite burden). Then $\mathcal{K}$ is strong (resp. of finite burden).*

- But the bound is given by some Ramsey number!

Theorem

[Ch., Simon] All ultraproducts of p-adics are inp-*minimal.*

Fact

[Simon] Let G be inp-*minimal. Then there is a definable normal abelian subgroup H such that G/H is of finite exponent.*

- **Question**: What happens in higher dimensions? Is burden subadditive, at least in this example?

Burden of VFA_0

- What is the burden of VFA_0? We know that it is bounded.
- **Observation**: [Ch.,Hils] Burden of VFA_0 is $\geq n$ for all $n \in \omega$ (as every completion of ACFA has a 1-type of weight n).
- **Problem**: Is VFA_0 strong?

Algebraic implications of strength and finite burden

- Results about definable objects can be now proved about type-definable objects.
- **Proposition** [Ch., Kaplan, Simon], a slight generalization of the argument of [Krupinski, Pillay] for the stable case: Any infinite strong field is perfect.
- A valued field (K, v) of characteristic $p > 0$ is *Kaplansky* if it satisfies:
  1. The valuation group Γ is p-divisible.
  2. The residue field k is perfect, and does not admit a finite separable extension whose degree is divisible by p.

Corollary

[Ch., Kaplan, Simon] Every strongly dependent (i.e. strong and dependent) valued field is Kaplansky.

Conjecture about definable envelopes of groups

1. [Shelah], [Aldama] If G is a group definable in an NIP theory and H is a subgroup which is abelian (nilpotent of class n; normal and soluble of derived length n) then there is a definable group containing H which is also abelian (resp. nilpotent of class n; normal and soluble of derived length n).
2. [Milliet] Let G be a group definable in a simple theory and let H be a subgroup of G.
   2.1 If H is nilpotent of class n, then there is a definable (with parameters from H) nilpotent group of class at most $2n$ finitely many translates of which cover H. If H is in addition normal, then there is a definable normal nilpotent group of class at most $3n$ containing H.
   2.2 If H is a soluble of class n, then there is a definable (with parameters from H) soluble group of derived length at most $2n$ finitely many translates of which cover H. If H is in addition normal, then there is a definable normal soluble group of derived length at most $3n$ containing H.

Conjecture about definable envelopes of groups

Conjecture: Let G be an NTP_2 group and assume that H is a subgroup. If H is nilpotent (soluble), then there is a definable nilpotent (resp. soluble) group finitely many translates of which cover H. If H is in addition normal, then there is a definable normal nilpotent (resp. soluble) group containing H.

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