External definability and groups in NIP

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Setting

- T is a complete first-order theory in a language L, countable for simplicity.
- $\mathbb{M} \models T$ — a monster model, κ-saturated for some sufficiently large cardinal κ.
- G — a group definable over $\emptyset$.
- As usual, for any set A we denote by $S(A)$ the (compact, Hausdorff) space of types over A and by $S_G(A)$ the set of types in G.

NIP

- A formula $\phi(x, y)$ has IP, the independence property, if in $\mathbb{M}$ there are tuples $(a_i)_{i \in \omega}$ and $(b_s)_{s \subseteq \omega}$ such that $\mathbb{M} \models \phi(a_i, b_s) \Leftrightarrow i \in s$.
- T is NIP if no formula has IP.
- Examples of NIP theories:
  - stable theories (e.g. modules, algebraically/separably/differentially closed fields, free groups),
  - ordered abelian groups,
  - o-minimal theories (real closed fields with exponentiation, etc.),
  - algebraically closed valued fields, p-adics.

Externally definable sets

- Given $M \models T$, we say that $X \subseteq M$ is *externally definable* if $X = M \cap \phi(x, a)$ for some $\phi \in L$ and $a \in \mathbb{M}$.
- T is stable if and only if for *every* model M, all of its externally definable subsets are M-definable (i.e. all types over all models are definable).
- *Some* unstable models also satisfy this property: $(\mathbb{R}, +, \times, 0, 1)$, $(\mathbb{Q}_p, +, \times, 0, 1)$, $(\mathbb{Z}, <, +, 0, 1)$, any maximally complete model of ACVF with $\mathbb{R}$ as its value group.
- Not true in NIP in general (consider $(\mathbb{Q}, <)$ and $X = \{x < \sqrt{2}\}$).

Externally definable sets in NIP

- However, externally definable sets in NIP demonstrate some tame behavior (and admit certain approximations in terms of internally definable sets).
- [Shelah] Let T be NIP, $M \models T$. Consider an expansion M^{ext} of M in the language L' with a new predicate symbol added for every externally definable subset of M^n. Then $\text{Th}_{L'}(M^{\text{ext}})$ eliminates quantifiers (i.e. a projection of an externally definable subset is externally definable), and is NIP.
- So we can make all types over a fixed model definable by passing to the corresponding Shelah's expansion. But which properties of definable groups are preserved under this operation?

Definable amenability

- An M-definable group $G(M)$ is called *definably amenable* if there is a finitely additive probability measure μ on M-definable subsets of G which is $G(M)$-invariant (say, on the left).
- Equivalently, there is a Borel probability measure on $S_G(M)$ which is invariant under the natural action of G on the space of types.
- This is an elementary property: $G(M)$ is def. amenable and $N \succeq M$ implies that $G(N)$ is def. amenable.

Definable amenability: examples

- Examples of definably amenable groups:
  - amenable groups (so e.g. solvable groups),
  - stable groups (so $\mathbb{F}_2$ is def. amenable but not amenable),
  - def. compact groups in *o*-minimal theories.
- Non-examples:
  - K is a saturated algebraically closed valued field or a real closed field and $n > 1$, then $SL(n, K)$ is not definably amenable.

Definable amenability in Shelah's expansion

Theorem

Assume that T is NIP, $M \models T$ *and* G *is def. amenable. Then* G *is still def. amenable in the sense of* M^{ext}*: there is a Borel probability measure* μ' *on* $S_G(M^{\text{ext}})$*, extending* μ *and* $G(M)$*-invariant.*

- Also holds for *definable extreme amenability*, i.e. the existence of a $G(M)$-invariant type.

Definable amenability in Shelah's expansion

Sketch of proof in the case of types: So let $p \in S_G(M)$ be fixed by $G(M)$.

1. [Ch.-Kaplan] If T is NIP, then there is a global type $p' \supseteq p$ which is both invariant over M and an heir over M. Since p' is an heir of p, it is still $G(M)$-invariant.
2. [Simon] In NIP, there is a continuous retraction F_M from the space of global M-invariant types onto the space of global types finitely satisfiable in M, which commutes with M-definable maps (follows from the proof of existence of honest definitions). Let $p'' = F_M(p')$, then p'' is finitely satisfiable in M and is still $G(M)$-invariant.
3. Finally, a type in $S(M^{\text{ext}})$ is the same thing as a type in $S(\mathbb{M})$ which is finitely satisfiable in M.

For the general case we generalize each of the steps to Keisler measures (using that measures in NIP are approximable by the averages of finite families of types, etc).

Model-theoretic connected components

Let A be a small subset of $\mathbb{M}$. We define:

- $G_A^0 = \bigcap \{H \leq G : H \text{ is } A\text{-definable, of finite index}\}$.
- $G_A^{00} =$
  $\bigcap \{H \leq G : H \text{ is type-definable over } A\text{, of bounded index, i.e. } < \kappa\}$.
- $G_A^{\infty} =$
  $\bigcap \{H \leq G : H \text{ is } \mathrm{Aut}(\mathbb{M}/A)\text{-invariant, of bounded index}\}$.
- Of course $G_A^0 \supseteq G_A^{00} \supseteq G_A^{\infty}$, and in general all these subgroups get smaller as A grows.

Connected components in NIP

Let T be NIP. Then for every small set A we have:

- $G^0_\emptyset = G^0_A$
- [Shelah] $G^{00}_\emptyset = G^{00}_A$.
- [Shelah for abelian groups, Gismatullin in general] $G^\infty_\emptyset = G^\infty_A$.
- All these are normal subgroups of G of finite (resp. bounded) index. We will be omitting $\emptyset$ in the subscript.

f-generic types

- A type $p \in S(\mathbb{M})$ is *f-generic over M* if $g \cdot p$ is $\mathrm{Aut}(\mathbb{M}/M)$-invariant for every $g \in G(\mathbb{M})$.
- "f" is for forking, which coincides with non-invariance over models of NIP theories.
- [Hrushovski, Pillay] Assuming NIP, $G(\mathbb{M})$ is definably amenable iff there is a global type which is f-generic over some (equivalently any) small model $M \prec \mathbb{M}$.

Connected components in NIP

- [Conversano, Pillay] There are NIP groups in which $G^{00} \neq G^{\infty}$.
- If p is f-generic, then $\mathrm{Stab}(p) = G^{00} = G^{\infty}$ (where $\mathrm{Stab}(p) = \{g \in G : g \cdot p = p\}$).
- So in particular if G is definably amenable, then $G^{00} = G^{\infty}$.

Connected components in Shelah's expansion

- What happens to these connected components if we consider G as a definable group in M^{ext}?
- In general an externally definable subgroup of $G(M)$ (i.e. an externally definable subset of $G(M)$ which happens to be a subgroup) need not contain any internally definable subgroups:

Example

Let $M \succ (\mathbb{R}, +, \cdot)$ be $\left(2^{\aleph_0}\right)^+$-saturated. Then M contains a subgroup $H = \left\{x \in M : \bigwedge_{r \in \mathbb{R}} |x| < r\right\}$ of infinitesimal elements. Note that H is externally definable as $M \cap c < x < d$ where $c, d \in \mathbb{M}$ realize the appropriate cuts of M. However H does not contain any M-definable subgroups as by o-minimality any such group would have to be a union of finitely many intervals in M.

Connected components in Shelah's expansion

Theorem

Let T be an NIP *theory in a language L, and $M \models T$. Let $T' = \mathrm{Th}\left(M^{\mathrm{ext}}\right)$, and let $\mathbb{M}'$ be a monster model of T'. Let $\mathbb{M} = \mathbb{M}' \restriction L$ — a monster model of T. Then we have:*

1. $G^{0}\left(\mathbb{M}\right) = G^{0}\left(\mathbb{M}'\right)$
2. $G^{00}\left(\mathbb{M}\right) = G^{00}\left(\mathbb{M}'\right)$
3. $G^{\infty}\left(\mathbb{M}\right) = G^{\infty}\left(\mathbb{M}'\right)$.

Corollary

Let T be NIP *and let M be a model. Assume that G is an externally definable subgroup of M of finite index. Then it is internally definable.*

Connected components in Shelah's expansion

- For the proof we first establish existence of the corresponding connected components in NIP relatively to a predicate and a sublanguage, and then we show that this relative connected components coincide with the connected components of the theory induced on the predicate, by a certain "catch-your-own-tail" construction of a chain of saturated pairs of models.

Definable topological dynamics

- [Newelski], [Pillay]
- Setting: T is NIP, $M \models T$, G is an M-definable group, $M_0 = M^{\text{ext}}$ (so all types over M_0 are definable).
- G acts naturally on $S_G(M_0)$ by homeomorphisms, the orbit of 1 is dense.
- $S_G(M_0)$ has a semigroup structure $\cdot$ extending the group operation on $G(M_0)$, continuous in the first coordinate: for $p, q \in S_G(M_0)$, $p \cdot q$ is $\text{tp}(a \cdot b / M_0)$ where $b \models q$ and a realizes the unique coheir of of p over M_0, b.
- There is a unique (up to isomorphism) minimal subflow of $S_G(M_0)$ (a subflow is a closed subset which is $G(M_0)$-invariant, equivalently a left ideal of the semigroup $S_G(M_0)$).
- Pick a minimal subflow $\mathcal{M}$, then there is an idempotent $u \in \mathcal{M}$. Then $u \cdot \mathcal{M}$ is a subgroup of the semigroup $S_G(M_0)$, and its isomorphism type does not depend on the choice of $\mathcal{M}$ and u. We call this the Ellis group (attached to the data).

Ellis group conjecture

- The canonical surjective homomorphism $G \to G/G^{00}$ factors naturally through the space $S_G(M^{\text{ext}})$, namely we have a well defined cont. surjection $\pi : S_G(M_0) \to G/G^{00}$, $\operatorname{tp}(g/M) \mapsto gG^{00}$ and $\pi \restriction u \cdot \mathcal{M}$ is a surjective group homomorphism.
- Newelski had suggested that for NIP groups, the Ellis group should be closely related (or even isomorphic) to G/G^{00}.
- [Gismatullin, Penazzi, Pillay] Fails for $SL(2, \mathbb{R})$ (if K is a saturated real closed field then $G^{00}(K) = G(K)$, but $u \cdot M$ is non-trivial).
- **Corrected conjecture**: Suppose G is definably amenable, NIP. Then the restriction of $\pi : S_G(M_0) \to G/G^{00}$ to $u \cdot \mathcal{M}$ is an isomorphism (for some/any choice of a minimal subflow $\mathcal{M}$ of $S_G(M_0)$ and an idempotent $u \in \mathcal{M}$).

Some new cases of the Ellis group conjecture

Theorem

The Ellis group conjecture for definably amenable groups is true in the following cases:

1. *[Pillay] G is fsg (i.e. when there is some $p \in S_G(\mathbb{M})$ and small $M \prec \mathbb{M}$ such that $g \cdot p$ is finitely satisfiable in M for all $g \in G$).*
2. *G has a definable f-generic (i.e. when there is some $p \in S_G(\mathbb{M})$ which is f-generic over M and definable over M).*
3. *G is dp-minimal (every such group is either fsg or has a definable f-generic).*
4. *G is a definably amenable group in an o-minimal theory (as every such group is "(definable f-generic)-by-fsg").*