Strong Erdős-Hajnal property in model theory

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Strong Erdős-Hajnal property

- Let U, V be infinite sets and $E \subseteq U \times V$ a bipartite graph.

Definition

We say that E satisfies the *Strong Erdős-Hajnal property*, or Strong EH, if there is $\delta \in \mathbb{R}_{>0}$ such that for any finite $A \subseteq U, B \subseteq V$ there are some $A_0 \subseteq A, B_0 \subseteq B$ with $|A_0| \geq \delta |A|, |B_0| \geq \delta |B|$ such that the pair (A_0, B_0) is *E-homogeneous*, i.e. either $(A_0 \times B_0) \subseteq E$ or $(A_0 \times B_0) \cap E = \emptyset$.

- We will be concerned with the case where $\mathcal{M}$ is a first-order structure, $U = M^{d_1}, V = M^{d_2}$ and $E \subseteq M^{d_1} \times M^{d_2}$ is definable in $\mathcal{M}$.

Fact

[Ramsey + Erdős] With no assumptions on E, one can find a homogeneous pair of subsets of logarithmic size, and it is the best possible (up to a constant) in general.

Corollary. If E satisfies strong EH, then E is NIP.

Examples with strong EH

- [Alon, Pach, Pinchasi, Radoičić, Sharir] Let $E \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ be semialgebraic. Then E satisfies strong EH.
- [Basu] Let E be a closed, definable relation in an *o*-minimal expansion of a field. Then E satisfies strong EH.

Theorem

*[C., Starchenko] Let $E(x, y)$ be definable in a distal structure. Then E satisfies **definable** strong EH, i.e. there are some $\delta \in \mathbb{R}_{>0}$ and formulas $\psi_1(x, z), \psi_2(y, z)$ such that for any finite $A \subseteq M^{|x|}, B \subseteq M^{|y|}$ there is some $c \in M^{|z|}$ such that the pair $A_0 := \psi(A, c), B_0 := \psi_2(B, c)$ is E-homogeneous with $|A_0| \geq \delta |A|, |B_0| \geq \delta |B|$.*
Moreover, if every binary relation definable in $\mathcal{M}$ satisfies definable strong EH, then $\mathcal{M}$ is distal.

- Examples of distal theories:
  - [Hrushovski, Pillay, Simon], [Simon] *o*-minimal theories, $\mathbb{Q}_p$.
  - [Aschenbrenner, C.] transseries, ($\approx$) OAG's, some valued fields.
  - [Boxall, Kestner] T is distal $\iff$ T^{Sh} is distal.

Reducts of distal theories and strong EH

- We say that a structure $\mathcal{M}$ satisfies strong EH if every relation definable in $\mathcal{M}$ satisfies strong EH.
- If $\mathcal{M}$ satisfies strong EH, then any structure interpretable in $\mathcal{M}$ also satisfies strong EH.
- E.g., ACF_0 satisfies strong EH — as $(\mathbb{C}, \times, +)$ is interpretable in a distal structure $(\mathbb{R}, \times, +)$.
- On the other hand, ACF_p doesn't!

ACF_p doesn't satisfy strong EH

Example

[C., Starchenko]

- Let $\mathcal{K} \models \mathrm{ACF}_p$.
- For a finite field $\mathbb{F}_q \subseteq \mathcal{K}$, where q is a power of p, let P_q be the set of all points in $\mathbb{F}_q^2$ and let L_q be the set of all lines in $\mathbb{F}_q^2$.
- Note $|P_q| = |L_q| = q^2$.
- Let $I \subseteq P_q \times L_q$ be the incidence relation. One can check:
- **Claim**. For any fixed $\delta > 0$, for all large enough q, if $L_0 \subseteq L_q$ and $P_0 \subseteq P_q$ with $|P_0| \geq \delta q^2$ and $|L_0| \geq \delta q^2$ then $I(P_0, L_0) \neq \emptyset$.
- As every finite field of char p can be embedded into $\mathcal{K}$, this shows that strong EH fails for the definable incidence relation $I \subseteq K^2 \times K^2$.

Local distality

- The difference between char 0 and char p is well-known in incidence combinatorics, and being a reduct of a distal structure (more precisely, admitting a distal cell decomposition, see below) appears to be a model-theoretic explanation for it.
- Our initial proof of strong EH in distal structures had a global assumption on the theory and gave non-optimal bounds.
- Under a global assumption of distality of the theory, a shorter (but even less informative in terms of the bounds) proof can be given (Simon, Pillay's talks).
- More recently, [C., Galvin, Starchenko] isolates a notion of local distality and provides a method to obtain good bounds.

Distal cell decomposition

- Let $E \subseteq U \times V$ and $\Delta \subseteq U$ be given.
- For $b \in V$, let $E(U, b) := \{a \in U : (a, b) \in E\}$.
- For $b \in V$, we say that $E(U, b)$ *crosses* Δ if $E(U, b) \cap \Delta \neq \emptyset$ and $\neg E(U, b) \cap \Delta \neq \emptyset$.
- Δ is *E-complete* over $B \subseteq V$ if Δ is not crossed by any $E(U, b)$ with $b \in B$.
- A family $\mathcal{F}$ of subsets of U is a *cell decomposition for E over B* if $U \subseteq \bigcup \mathcal{F}$ and every $\Delta \in \mathcal{F}$ is E-complete over B.
- A *cell decomposition for E* is an assignment $\mathcal{T}$ s.t. for each finite $B \subseteq V$, $\mathcal{T}(B)$ is a cell decomposition for E over B.
- A cell decomposition $\mathcal{T}$ is *distal* if for some $k \in \mathbb{N}$ there is a relation $D \subseteq U \times V^k$ s.t. all finite $B \subseteq V$, $\mathcal{T}(B) = \{D(U; b_1, \ldots, b_k) : b_1, \ldots, b_k \in B \text{ and } D(U; b_1, \ldots, b_k) \text{ is } E\text{-complete over } B\}$.
- A relation E is *distal* if it admits a distal cell decomposition.

Example

1. E is distal $\implies$ E is NIP (the number of E-types over any finite set B is at most $|B|^k$)
2. Any relation definable in a reduct of a distal structure admits a distal cell decomposition (follows from the existence of strong honest definitions in distal theories [C., Simon]).

Theorem

[C., Galvin, Starchenko] Le $\mathcal{M}$ be an o-minimal expansion of a field and let $E(x, y)$ with $|x| = 2$ be definable. Then $E(x, y)$ admits a distal cell decomposition $\mathcal{T}$ with $|\mathcal{T}(S)| = O\left(|S|^2\right)$ for all finite sets S.

- In higher dimensions, becomes much more difficult to obtain an optimal bound, even in the semialgebraic case.

Cutting

- So called cutting lemmas are a very important "divide and conquer" method for counting incidences in geometric combinatorics.

Theorem

[C., Galvin, Starchenko] (Distal cutting lemma) Assume $E(x,y) \subseteq M^{|x|} \times M^{|y|}$ admits a distal cell decomposition $\mathcal{T}$ with $|\mathcal{T}(S)| = O\left(|S|^d\right)$ for all finite sets $S \subseteq M^{|y|}$. Then there is a constant c s.t. for any finite $S \subseteq M^{|y|}$ of size n and any real $1 < r < n$, there is a covering $X_1, \ldots, X_t$ of $M^{|x|}$ with $t \leq cr^d$ and each X_i crossed by at most $\frac{n}{r}$ of the sets $\{E(x,b) : b \in S\}$.

Applications of cuttings

1. Assume $E \subseteq U \times V$ satisfies the conclusion of the cutting lemma. Then it satisfies strong EH.
2. (o-minimal generalization of the Szemeredi-Trotter theorem) Let $\mathcal{M}$ be an o-minimal expansion of a field and $E(x, y) \subseteq M^2 \times M^2$ definable. Then for any $k \in \omega$ there is some $c \in \mathbb{R}_{>0}$ satisfying the following: for any $A, B \subseteq M^2$, if $E(A, B)$ is $K_{k,k}$-free, then $|E(A, B)| \leq cn^{\frac{4}{3}}$.
   [Fox, Pach, Sheffer, Suk, Zahl] in the semialgebraic case, [Basu, Raz] under a stronger assumption.
3. An ε-version of the Elekes-Szabó theorem.
4. Etc.

1-based theories

- ACF_p is the only known example of an NIP theory not satisfying strong EH (as well as the only example without a distal expansion).
- Zilber's trichotomy principle: roughly, every strongly minimal set is either like an infinite set, or like a vector space, or interprets a field.

Definition

("like a vector space")

1. A formula $E(x,y)$ is *weakly normal* if $\exists k \in \mathbb{N}$ s.t. the intersection of any k pairwise distinct sets of the form $E(M,b), b \in M^{|y|}$ is empty.
2. T is 1-based if every formula is a Boolean combination of weakly normal formulas.

- Note: this definition implies stability of T, and is equivalent to: for any small set A, B, $A \mathop{\smile\!\!\!\!\!\!\mid}_{\mathrm{acl}^{\mathrm{eq}}(A) \cap \mathrm{acl}^{\mathrm{eq}}(B)} B$.

1-based theories satisfy strong EH

- Main examples: abelian groups, modules.
- In a sense, these are the only examples:
- [Hrushovski, Pillay] Let $(G, \cdot, \ldots)$ be a 1-based group. Then all definable subset of G^n are Boolean combinations of cosets of $\emptyset$-definable subgroups of G^n.

Theorem

[C., Starchenko] Every stable 1*-based theory satisfies strong EH.*

- Problem reduces to showing strong EH for weakly normal formulas (using that weakly normal formulas are closed under conjunctions).
- Via some manipulations and basic linear algebra, the incidence problem for a k-weakly normal formula reduces to an incidence problem for an affine hyperplanes arrangement in $\mathbb{R}^k$.
- Which is definable in $\mathbb{R}$, hence has strong EH by distality.
- Somewhat curiously, we have to use RCF in a proof for a stable structure! (Again, typical in incidence combinatorics.)