Incidence counting and trichotomy in o-minimal structures

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Hypergraphs and Zarankiewicz's problem

- We fix $r \in \mathbb{N}_{\geq 2}$ and let $H = (V_1, \ldots, V_r; E)$ be an r-partite and r-uniform hypergraph (or just *r-hypergraph*) with vertex sets $V_1, \ldots, V_r$ with $|V_i| = n_i$, (hyper-) edge set $E \subseteq \prod_{i \in [r]} V_i$, and $n = \sum_{i=1}^{r} n_i$ is the total number of vertices.
- When $r = 2$, we say "bipartite graph" instead of "2-hypergraph".
- For $k \in \mathbb{N}$, let $K_{k,\ldots,k}$ denote the complete r-hypergraph with each part of size k (i.e. $V_i = [k]$ and $E = \prod_{i \in [k]} V_i$).
- H is *$K_{k,\ldots,k}$-free* if it does note contain an isomorphic copy of $K_{k,\ldots,k}$.
- Zarankiewicz's problem: for fixed r, k, what is the maximal number of edges $|E|$ in a $K_{k,\ldots,k}$-free r-hypergraph H? (As a functions of $n_1, \ldots, n_r$).

Number of edges in a $K_{k,\dots,k}$-free hypergraph

- The following fact is due to [Kővári, Sós, Turán'54] for $r = 2$ and [Erdős'64] for general r.

Fact (The Basic Bound)
If H is a $K_{k,\dots,k}$-free r-hypergraph then $|E| = O_{r,k}\left(n^{r-\frac{1}{k^{r-1}}}\right)$.

- "$= O_{r,k}(-)$" means "$\leq c \cdot -$" for some constant $c \in \mathbb{R}$ depending only on r and k.
- So the exponent is slightly better than the maximal possible r (we have n^r edges in $K_{n,\dots,n}$). A probabilistic construction in [Erdős'64] shows that it cannot be substantially improved.

Families of hypergraphs induced by definable relations

- Let $\mathcal{M} = (M, \ldots)$ be a first-order structure in a language $\mathcal{L}$, and let $R \subseteq M_{x_1} \times \ldots \times M_{x_r}$ be a definable relation on the product of some sorts of $\mathcal{M}$.
- We let $\mathcal{F}_R$ be the family of all finite r-hypergraphs induced by R, i.e. hypergraphs of the form

$$H = (V_1, \ldots, V_r; R \restriction_{V_1 \times \ldots \times V_r})$$

  for some finite $V_i \subseteq M_{x_i}, i \in [r]$.
- **Question.** What properties of the structure $\mathcal{M}$ are reflected by the Zarankiewicz-style bounds for the families of hypergraphs $\mathcal{F}_R$ with R definable in $\mathcal{M}$?

Point-line incidences, char p

- Let $K \models \mathrm{ACF}_p$ be an algebraically closed field of positive characteristic.
- Let $R \subseteq K^2 \times K^2$ be the (definable) incidence relation between points and lines in K^2, i.e.

$$R(x_1, x_2; y_1, y_2) \iff x_2 = y_1 x_1 + y_2.$$

- Note that R is $K_{2,2}$-free (there is a unique line through any two distinct points).
- Let q be a power of p, then $\mathbb{F}_q \subseteq K$ and we take $V_1 = V_2 = (\mathbb{F}_q)^2$ (i.e. the set of all points and the set of all lines in $\mathbb{F}_q^2$), $E = R \restriction_{V_1 \times V_2}$. Then $H = (V_1, V_2; E) \in \mathcal{F}_R$.
- We have $|V_1| = |V_2| = q^2$ and $|E| = q\,|V_2| = q^3$.
- Let $n := q^2$, then $|V_1| = |V_2| = n$ and $|E| \geq n^{\frac{3}{2}}$ — matches the Basic Bound for $r = k = 2$.

Points-lines incidences, char 0

- On the other hand, over the reals a bound strictly better than the Basic Bound holds ($\frac{4}{3} < \frac{3}{2}$):

Fact (Szémeredi-Trotter '83)

Let $R \subseteq \mathbb{R}^2 \times \mathbb{R}^2$ be the incidence relation between points and lines in $\mathbb{R}^2$. Then every $H \in \mathcal{F}_R$ satisfies $|E| = O\left(n^{\frac{4}{3}}\right)$.

- Known to be optimal up to a constant.
- In fact, the same holds in ACF_0:

Fact (Tóth '03)

Let $R \subseteq \mathbb{C}^2 \times \mathbb{C}^2$ be the incidence relation between points and lines in $\mathbb{C}^2$. Then every $H \in \mathcal{F}_R$ satisfies $|E| = O\left(n^{\frac{4}{3}}\right)$.

- Reason: ACF_0 is a reduct of a distal theory, while ACF_p is not.

Stronger bounds for hypergraphs definable in distal structures

- Generalizing a result of [Fox, Pach, Sheffer, Suk, Zahl'15] in the semialgebraic case, we have:

Fact (C., Galvin, Starchenko'16)

Let $\mathcal{M}$ be a distal structure and $R \subseteq M_{x_1} \times M_{x_2}$ a definable relation. Then there exists some $\varepsilon = \varepsilon(R, k) > 0$ such that every $K_{k,k}$-free bipartite graph $H \in \mathcal{F}_R$ satisfies $|E| = O_{R,k}(n^{t-\varepsilon})$, where t is the exponent given by the Basic Bound.

- In fact, ε is given in terms of k and the size of the smallest *distal cell decomposition* for R.
- E.g. if $R \subseteq M^2 \times M^2$ for an *o*-minimal $\mathcal{M}$, then $t - \varepsilon = \frac{4}{3}$ ([C., Galvin, Starchenko'16]; independently, [Basu, Raz'16]).
- Bounds for $R \subseteq M^{d_1} \times M^{d_2}$ with $\mathcal{M} \models \mathrm{RCF}$ [Fox, Pach, Sheffer, Suk, Zahl'15]; $\mathcal{M}$ is *o*-minimal [Anderson'20].

Connections to the trichotomy principle

- If $\mathcal{M}$ is sufficiently tame model-theoretically (e.g. stable/geometric + distal expansion; or more concretely, ACF_0 or *o*-minimal), the exponents in Zarankiewicz bounds appear to reflect the trichotomy principle, and detect presence of algebraic structures (groups, fields).
- Instances of this principle are well-known in combinatorics — extremal configuration for various counting problems tend to possess algebraic structure.

Example: detecting groups and Elekes-Szabó theorem

Fact (Elekes-Szabó'12)

Let $\mathcal{M} \models \mathrm{ACF}_0$ be saturated, X_1, X_2, X_3 strongly minimal definable sets, $R \subseteq X_1 \times X_2 \times X_3$ has Morley rank 2, and R is $K_{k,k}$-free under any partition of its variables into two groups. Then exactly one of the following holds.

(a) *For some $\varepsilon > 0$, $|E| = O(n^{2-\varepsilon})$ for every $H \in \mathcal{F}_R$.*

(b) *there exists a definable group G of Morley rank and degree 1, elements $g_i \in G, \alpha_i \in X_i$ with α_i and g_i inter-algebraic (over some set of parameters C) for $i \in [3]$, $\bar{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ is generic in R over C and $g_1 \cdot g_2 \cdot g_3 = 1$ in G.*

- Some more recent generalizations:
  - [Hrushovski'13];
  - [Bays-Breuillard'18] for ACF_0 and R of any arity;
  - [C., Starchenko'18] for $\mathcal{M}$ strongly minimal with a distal expansion, R of arity 3;
  - [C., Peterzil, Starchenko'20] $\mathcal{M}$ stable with distal expansion or o-minimal, R of any arity, codimension 1.
- Proofs combine "stronger than basic" Zarankiewicz bounds with variants of the group configuration theorem.
- In this talk — a new result showing that fields can be detected from the exponents, at least in o-minimal structures and working *globally* (i.e. working with all $\{\mathcal{F}_R : R \text{ definable}\}$ simultaneously rather with a single $\mathcal{F}_R$).
- Main new ingredient — even stronger Zarankiewicz bounds in locally modular structures.

An abstract setting: coordinate-wise monotone functions and basic relations

- Let $V = \prod_{i \in [r]} V_i$ and $(S, <)$ a linearly ordered set. A function $f \colon V \to S$ is *coordinate-wise monotone* if
  - for any $i \in [r]$,
  - any $a = (a_j)_{j \in [r] \setminus \{i\}}$, $a' = (a'_j)_{j \in [r] \setminus \{i\}} \in \prod_{j \neq i} V_j$,
  - and any $b, b' \in V_i$

  we have

$$f(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_r) \leq f(a_1, \ldots, a_{i-1}, b', a_{i+1}, \ldots, a_r)$$
$$\iff$$
$$f(a'_1, \ldots, a'_{i-1}, b, a'_{i+1}, \ldots, a'_r) \leq f(a'_1, \ldots, a'_{i-1}, b', a'_{i+1}, \ldots, a'_r).$$

- A subset $X \subseteq V$ is *basic* if there exists a linearly ordered set $(S, <)$, a coordinate-wise monotone function $f \colon V \to S$ and $\ell \in S$ such that $X = \{b \in V \colon f(b) < \ell\}$.
- A set $X \subseteq V$ has *grid complexity* $\leq s$ if X is an intersection of V with at most s basic subsets of V.

Example: semilinear relations of bounded complexity

- Let W be an ordered vector space over an ordered division ring R. A set $X \subseteq W^d$ is *semilinear* if X is a finite union of sets of the form

  $$\left\{ \bar{x} \in W^d : f_1(\bar{x}) \leq 0, \ldots, f_p(\bar{x}) \leq 0, f_{p+1}(\bar{x}) < 0, \ldots, f_q(\bar{x}) < 0 \right\},$$

  where $p \leq q \in \mathbb{N}$ and each $f_i : V^d \to V$ is a *linear* function

  $$f(x_1, \ldots, x_d) = \lambda_1 x_1 + \ldots + \lambda_d x_d + a$$

  for some $\lambda_i \in R$ and $a \in V$.
- Note that every linear function f is coordinate-wise monotone.
- Hence, if $d = d_1 + \ldots + d_r$, $X \subseteq W^d = \prod_{i \in [r]} W^{d_i}$ is of grid complexity q.

Zarankiewicz bound for relations of bounded grid complexity

Theorem

For every integers $r \geq 2, s \geq 0, k \geq 2$ there are $\alpha = \alpha(r, s, k) \in \mathbb{R}$ and $\beta = \beta(r, s) \in \mathbb{N}$ such that: for any finite $K_{k,\ldots,k}$-free r-hypergraph $H = (V_1, \ldots, V_r; E)$ with $E \subseteq \prod_{i \in [r]} V_i$ of grid complexity $\leq s$ we have

$$|E| \leq \alpha n^{r-1} (\log n)^{\beta} .$$

Moreover, we can take $\beta(r, s) := s(2^{r-1} - 1)$.

- In particular, $|E| = O_{r,s,k,\varepsilon}(n^{r-1+\varepsilon})$ for any $\varepsilon > 0$.
- Our proof is by double recursion on the grid complexity and the complexities of certain derived hypergraphs of smaller arity, coordinate-wise monotone maps into linear orders are used in the recursive step to pick the "middle point" splitting the vertices in a balanced way.

Corollary for semilinear hypergraphs

Corollary

For every $s, k \in \mathbb{N}$ *there exist some* $\alpha = \alpha(r, s, k) \in \mathbb{R}$ *and* $\beta(r, s) := s(2^{r-1} - 1)$ *satisfying the following.*
Suppose that $r \geq 2, d = d_1 + \ldots + d_r \in \mathbb{N}$ *and* $R \subseteq \mathbb{R}^{d_1} \times \ldots \times \mathbb{R}^{d_r}$ *is semilinear and defined by* $\leq s$ *linear equalities and inequalities.*
Then for every $K_{k,\ldots,k}$*-free* r*-hypergraph* $H \in \mathcal{F}_R$ *we have*

$$|E| \leq \alpha n^{r-1} (\log n)^{\beta} .$$

An application to incidences with polytopes, 1

- Applying with $r = 2$ we get the following:

Corollary

For every $s, k \in \mathbb{N}$ there exists some $\alpha = \alpha(s, k) \in \mathbb{R}$ satisfying the following.
Let $d \in \mathbb{N}$ and $H_1, \ldots, H_q \subseteq \mathbb{R}^d$ be finitely many (closed or open) half-spaces in $\mathbb{R}^d$. Let $\mathcal{F}$ be the (infinite) family of all polytopes in $\mathbb{R}^d$ cut out by arbitrary translates of $H_1, \ldots, H_q$.
For any set V_1 of n_1 points in $\mathbb{R}^d$ and any set V_2 of n_2 polytopes in $\mathcal{F}$, if the incidence graph on $V_1 \times V_2$ is $K_{k,k}$-free, then it contains at most $\alpha n (\log n)^q$ incidences.

An application to incidences with polytopes, 2

- In particular (much better than the general semialgebraic bound):

Corollary

For any set V_1 of n_1 points and any set V_2 of n_2 (solid) boxes with axis parallel sides in $\mathbb{R}^d$, if the incidence graph on $V_1 \times V_2$ is $K_{k,k}$-free, then it contains at most $O_{d,k}\left(n(\log n)^{2d}\right)$ incidences.

- Independently, a similar bound for the intersection graphs of boxes [Tomon, Zakharov'20].

Dyadic rectangles and a lower bound

- Is the logarithmic factor necessary?
- We focus on the simplest case of incidences with rectangles with axis-parallel sides in $\mathbb{R}^2$. The previous corollary gives the bound $O_{d,k}\left(n(\log n)^4\right)$.
- A box is *dyadic* if it is the direct products of intervals of the form $[s2^t, (s+1)2^t)$ for some integers s, t.
- Using a different argument, restricting to dyadic boxes we get a stronger upper bound $O\left(n\frac{\log n}{\log\log n}\right)$, and give a construction showing a matching lower bound (up to a constant).
- [Tomon, Zakharov'20] get the upper bound $O_{d,k}\left(n(\log n)\right)$ in the $K_{2,2}$-free case, and use our lower bound construction to provide a counterexample to a conjecture of [Alon, Basavaraju, Chandran, Mathew, Rajendraprasad, 15] about the number of edges in a graph of bounded "separation dimension".

Problem

Does the power of $\log n$ *have to grow with the dimension* d*?*

Geometric weakly locally modular theories

- In our bounds, we can get rid of the logarithmic factor entirely restricting to the family of all finite r-hypergraphs induced by a given $K_{k,\dots,k}$-free relation (as opposed to all $K_{k,\dots,k}$-free r-hypergraphs induced by a given relation).
- Recall that a complete first-order theory T is *geometric* if, in any model $\mathcal{M} \models T$, the algebraic closure operator satisfies the *Exchange Principle* and the quantifier $\exists^\infty$ is eliminated.
- Hence, in a model of a geometric theory, acl defines a well-behaved notion of independence $\mathop{\smile\!\!\!\!\!\!|}\,$.
- [Berenstein, Vassiliev] A geometric theory is *weakly locally modular* if for any small subsets $A, B \subseteq \mathbb{M} \models T$ there exists some small set $C \mathop{\smile\!\!\!\!\!\!|}_{\emptyset} AB$ such that $A \mathop{\smile\!\!\!\!\!\!|}_{\operatorname{acl}(AC) \cap \operatorname{acl}(BC)} B$.
- E.g. any o-minimal theory T is geometric, and T is weakly locally modular if and only if T is linear (i.e. any normal interpretable family of plane curves in T has dimension ≤ 1).

Bound for $K_{k,\ldots,k}$-free relations in geometric weakly locally modular structures

Theorem

Assume that T is a geometric, weakly locally modular theory, and $\mathcal{M} \models T$. Assume that $r \in \mathbb{N}_{\geq 2}$ and $R \subseteq M_{x_1} \times \ldots \times M_{x_r}$ is definable and $K_{k,\ldots,k}$-free. Then for every $H \in \mathcal{F}_R$ we have

$$|E| = O_R(n^{r-1}).$$

Moreover, if T is distal, then can relax "$K_{k,\ldots,k}$-free" to "does not contain the direct product of r infinite sets".

A related observation was made by Evans in the binary case for certain stable theories.

Recovering a field in the *o*-minimal case

Fact (Peterzil, Starchenko'98)

Let $\mathcal{M}$ be an o-minimal saturated structure. TFAE:

1. *$\mathcal{M}$ is not weakly locally modular;*
2. *there exists a real closed field definable in $\mathcal{M}$.*

- Combining this with the previous theorem, we thus get:

Corollary

Let $\mathcal{M}$ be an o-minimal structure. TFAE:

1. *$\mathcal{M}$ is weakly locally modular;*
2. *for every definable $K_{k,\ldots,k}$-free r-ary relation R, every $H \in \mathcal{F}_R$ satisfies $|E| = O(n^{r-1})$.*
3. *for every definable binary relation R, if all $H \in \mathcal{F}_R$ satisfy $|E| = O(n^{2-\varepsilon})$ for some $\varepsilon > 0$, then in fact $|E| = O(n)$;*
4. *no infinite field is definable in $\mathcal{M}$.*