Regularity for slice-wise stable hypergraphs

Artem Chernikov

UCLA

Beijing Logic Meeting 2023

Beijing, China, Jul 2023

- Joint work with Henry Towsner.

Context: ultraproducts of finite graphs with Loeb measure

- For each $i \in \mathbb{N}$, let $G_i = (V_i, E_i)$ be a graph with $|V_i|$ finite and $\lim_{i\to\infty} |V_i| = \infty$.
- Given a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, the ultraproduct

$$(V, E) = \prod_{i\in\mathbb{N}} (V_i, E_i)$$

  is a graph on the set V of size continuum.
- Given $k \in \mathbb{N}$ and an *internal* set $X \subseteq V^k$ (i.e. $X = \prod_{\mathcal{U}} X_i$ for some $X_i \subseteq V_i^k$), we define $\mu^k(X) := \lim_{\mathcal{U}} \frac{|X_i|}{|V_i|^k}$. Then:
  - μ^k is a finitely additive probability measure on the Boolean algebra of internal subsets of V^k,
  - extends uniquely to a countably additive measure on the σ-algebra $\mathcal{B}_k$ generated by the internals subsets of V^k.

Approximation by rectangles

- Let $\mathcal{B}_1 \times \mathcal{B}_1$ be the *product σ-algebra*, i.e. for every $E \in \mathcal{B}_1 \times \mathcal{B}_1$ and $\varepsilon > 0$ there exist $A_i, B_i \in \mathcal{B}_1$, $i < k$, so that

$$\mu^2 \left(E \Delta \left(\bigcup_{i<k} A_i \times B_i \right) \right) < \varepsilon.$$

- Note: $\mathcal{B}_1 \times \mathcal{B}_1 \subsetneq \mathcal{B}_2$ (e.g. for $E = \prod_{\mathcal{U}} E_i$ with E_i a uniformly random graph on V_i we have $E \in \mathcal{B}_2 \setminus (\mathcal{B}_1 \times \mathcal{B}_1)$).

Szemerédi's regularity lemma as a measure-theoretic statement: Elek-Szegedy, Tao, Towsner, ...

- [Szemerédi's regularity lemma] Given $E \in \mathcal{B}_2$ and $\varepsilon > 0$, there is a decomposition of the form

$$1_E = f_{\mathrm{str}} + f_{\mathrm{qr}} + f_{\mathrm{err}},$$

where:

  - $f_{\mathrm{str}} = \sum_{i \leq n} d_i 1_{A_i}(x) 1_{B_i}(y)$ for some $n \in \mathbb{N}$, $A_i, B_i \in \mathcal{B}_1$ and $d_i \in [0,1]$ (so f_{str} is $\mathcal{B}_1 \times \mathcal{B}_1$-simple),
  - $f_{\mathrm{err}} : V^2 \to [-1,1]$ and $\int_{V^2} |f_{\mathrm{err}}|^2 \, d\mu^2 < \varepsilon$,
  - f_{qr} is *quasi-random:* for any $A, B \in \mathcal{B}_1$ we have $\int_{V^2} 1_A(x) 1_B(y) f_{\mathrm{qr}}(x,y) \, d\mu^2 = 0$.

- Under what conditions on E can the quasi-random part be omitted?

VC-dimension

- Given $E \subseteq V^2$ and $x \in V$, let $E_x = \{y \in V : (x, y) \in E\}$ be the x-fiber of E.
- A graph $E \subseteq V^2$ has *VC-dimension* $\geq d$ if there are some $y_1, \ldots, y_d \in V$ such that, for every $S \subseteq \{y_1, \ldots, y_d\}$ there is $x \in V$ so that $E_x \cap \{y_1, \ldots, y_d\} = S$.
- **Example.** If E_i is a random graph on V_i and $(V, E) = \prod_{\mathcal{U}} (V_i, E_i)$, then $\mathrm{VC}(E) = \infty$.
- **Example.** If E is definable in an NIP theory (e.g. E is semialgebraic), then $\mathrm{VC}(E) < \infty$.

Regularity lemma for graphs of finite VC-dimension

- [Alon, Fischer, Newman] [Lovasz, Szegedy] [Hrushovski, Pillay, Simon] If $E \in \mathcal{B}_2$ and $\mathrm{VC}(E) < \infty$, then:
  - $E \in \mathcal{B}_1 \times \mathcal{B}_1$,
  - the number of rectangles needed to approximate E within ε is bounded by a polynomial in $\frac{1}{\varepsilon}$.

Hypergraph regularity

- We discuss 3-hypergraphs for simplicity.
- We have $\mathcal{B}_3 \supsetneq \mathcal{B}_1 \times \mathcal{B}_1 \times \mathcal{B}_1, \mathcal{B}_2 \times \mathcal{B}_1$, etc.
- Moreover, let $\mathcal{B}_{3,2} \subseteq \mathcal{B}_3$ be the σ-algebra generated by intersections of "cylindrical" sets of the form

$$\left\{(x,y,z) \in V^3 : (x,y) \in A \wedge (x,z) \in B \wedge (y,z) \in C\right\}$$

  for some $A, B, C \in \mathcal{B}_2$. Again, $\mathcal{B}_{3,2} \subsetneq \mathcal{B}_3$.
- [Hypergraph regularity lemma] Any $E \in \mathcal{B}_3$ can be decomposed as
  $1_E \approx f(x,y,z) + \sum_{i \leq m} \alpha_i 1_{A_i}(x,y) 1_{B_i}(x,z) 1_{C_i}(y,z) + \sum_{j \leq n} \beta_i 1_{D_i}(x) 1_{F_i}(y) 1_{G_i}(z)$,
  where f quasi-random w.r.t. $\mathcal{B}_{3,2}$, and $A_i, B_i, C_i \in \mathcal{B}_2$ are quasi-random w.r.t $\mathcal{B}_1 \times \mathcal{B}_1$, and $D_i, F_i, G_i \in \mathcal{B}_1$.
- Apart from f, the rest is $\mathcal{B}_{3,2}$-measurable. Under what conditions E is "binary", i.e. the ternary quasi-random f can be omitted?
- [C., Townser] Iff VC_2-dimension is finite.

Hypergraph regularity for hypergraphs of slice-wise finite VC-dimension

- Today we discuss the most restrictive case of measurability with respect to unary sets:
- Moreover, let $\mathcal{B}_{3,1} \subseteq \mathcal{B}_3$ be the σ-algebra generated by intersections of "cylindrical" sets of the form

$$\{(x, y, z) \in V^3 : x \in A \wedge y \in B \wedge z \in C\}$$

  for some $A, B, C \in \mathcal{B}_1$. Note: $\mathcal{B}_{3,1} \subsetneq \mathcal{B}_{3,2}$.
- $E \in \mathcal{B}_3$ has *slice-wise* finite VC-dimension if for (almost) every $b \in V$, the (binary) fiber $E_b = \{(x, y) \in V^2 : (x, y, b) \in E\} \in \mathcal{B}_2$ has finite VC-dimension (and the same for any permutation of the variables).
- [C., Starchenko] + [C., Towsner] $E \in \mathcal{B}_3$ is slice-wise finite VC-dimension iff $E \in \mathcal{B}_{3,1}$.

Stability and μ-stability

- Fix $E \in \mathcal{B}_2$.
- A *ladder for E of height d* is a tuple $\bar{a}^\frown \bar{b} = (a_i : i \in d)^\frown (b_i : i \in d)$ with $a_i \in V, b_i \in V$ such that for every $i, j \in d$ we have $(a_i, b_j) \in E \iff i \leq j$.
- E is *d-stable* if there are no ladders of height d for E, and *stable* if it is ladder d-stable for some $d \in \omega$.
- For regularity lemmas, we can ignore measure 0 ladders, so it is natural to relax the definition as follows:
- A *μ-ladder for E of height d* is a tuple $\bar{b} = (b_j : j \in d)$ so that for every $i \in d$ we have $\mu\left(\bigcap_{i \leq j} E_{b_j} \setminus \left(\bigcup_{j > i} E_{b_j}\right)\right) > 0$.
- For $E \in \mathcal{B}_2$, let $\mathrm{Lad}^{\mu,E,d} \in \mathcal{B}_d$ be the set of all $\bar{b} = (b_i : i \in d)$ so that $\bar{b}$ is a μ-ladder for E of height d.
- $E \in \mathcal{B}_2$ is *d-μ-stable* if $\mu\left(\mathrm{Lad}^{\mu,E,d}\right) = 0$. And E is *μ-stable* if it is ladder d-μ-stable for some $d \in \omega$.

Regularity for μ-stable graphs and hypergraphs

- A set $A \in \mathcal{B}_1$ is *perfect for* $E \in \mathcal{B}_2$ if $\mu\left(\{b \in V : \mu(E_b \cap A) > 0 \wedge \mu(A \setminus E_b) > 0\}\right) = 0$.
- Note: if $A, B \in \mathcal{B}_1$ are perfect for E, then $\frac{\mu(E \cap (A \times B))}{\mu(A \times B)} \in \{0, 1\}$.
- A simplified version of [Malliaris-Shelah]: Assume that $E \in \mathcal{B}_2$ is μ-stable. Then there exist countable partitions $V = \bigsqcup_{i \in \omega} A_i$ and $V = \bigsqcup_{j \in \omega} B_j$ into perfect sets. In particular, for each $i, j \in \omega$, $\frac{\mu\left(E \cap \left(A_i \times B_j\right)\right)}{\mu\left(A_i \times B_j\right)} \in \{0, 1\}$.
- What about hypergraphs?
- We say that $E \in \mathcal{B}_3$ is μ*-stable* if the binary relation $E(x; yz)$ is μ-stable, and the same for any other partition of the variables.
- [C.,Starchenko], [Ackerman, Freer, Patel] If $E \in \mathcal{B}_3$ is μ-stable, then there exist countable partitions A_i, B_j, C_k of V into perfect sets. In particular, for each $i, j, k \in \omega$, $\frac{\mu\left(E \cap \left(A_i \times B_j \times C_k\right)\right)}{\mu\left(A_i \times B_j \times C_k\right)} \in \{0, 1\}$.

Regularity for slice-wise μ-stable hypergraphs

- [Terry-Wolf] Does this also hold for slice-wise stable $E \in \mathcal{B}_3$?
- (This seems to be the last remaining question about measurability with respect to unary sets.)
- We say that $E \in \mathcal{B}_3$ is *slicewise μ-stable* if the binary fiber $E_b \in \mathcal{B}_2$ is μ-stable for almost all $b \in V$, and the same for every permutation of the coordinates.
- [C., Towsner] No! But we have the next best thing: Suppose that $E \in \mathcal{B}_3$ is slice-wise μ-stable. Then there exist countable partitions A_i, B_j, C_k of $V \times V$ so that: each A_i is perfect for the relation $E(xy; z)$, and $A^i = A^{i,X} \times A^{i,Y}$ is a rectangle with $A^{i,X}, A^{i,Y} \in \mathcal{B}_1$, and same for B_j, C_k with respect to the other partitions of the variables. In particular, for every i, j, k, $\frac{\mu\left(E(x,y,z) \wedge A_i(x,y) \wedge B_j(x,z) \wedge C_k(y,z)\right)}{\mu\left(A_i(x,y) \wedge B_j(x,z) \wedge C_k(y,z)\right)} \in \{0, 1\}$.

Idea of the proof

- So let $E \in X \times Y \times Z$ be slice-wise μ-stable.
- Then for (almost) every $x \in X$, $E_x \subseteq Y \times Z$ is μ-stable, so by the stable graph regularity can decompose Y, Z into perfect sets with respect to E_x. But a priori there is no relation between such decompositions of Y, Z for different x!
- To achieve uniformity, we are going to do a number of repartitions in a "definable" way.
- First, a general "symmetrization" result for binary relations:

Symmetrizing partitions for binary relations

Lemma

Assume $A \subseteq X \times Y$ with $A \in \mathcal{B}_{X\times Y}$. Then there exist countable partitions $X = \bigsqcup_{i\in\omega} U_i$ with $U_i \in \mathcal{B}_X$ and $Y = \bigsqcup_{i\in\omega} V_i$ with $V_i \in \mathcal{B}_Y$ such that for each $i \in \omega$ we have:

1. $\mu\left((A \cap (U_i \times Y)) \triangle (A \cap (X \times V_i))\right) = 0$,
2. *for any $U' \subseteq U_i, U' \in \mathcal{B}_X$ such that both $\mu(A \cap (U' \times Y)) > 0$ and $\mu(A \cap ((U_i \setminus U') \times Y)) > 0$, for any $V' \subseteq V_i, V' \in \mathcal{B}_Y$ we have $\mu\big((A \cap (U' \times Y)) \triangle (A \cap (U_i \times V'))\big) > 0$.*

In particular, A is almost contained in the rectangles on the diagonal, that is $\mu\left(A \setminus \bigcup_{i\in\omega}(U_i \times V_i)\right) = 0$.

Getting μ-stable graph regularity uniformly in fibers

As mentioned earlier, we have regularity for hypergraphs of slice-wise finite VC-dimension uniformly over fibers:

Lemma

Assume $E \in \mathcal{B}_{X \times Y \times Z}$ is such that for almost all $z \in Z$, the binary relation $E_z \in \mathcal{B}_{X \times Y}$ is μ-NIP. Then there exist $P^i \in \mathcal{B}^E_{X \times Z}, Q^i \in \mathcal{B}^E_{Y \times Z}$ for $i \in \omega$ such that for almost every $z \in Z$ we have $\chi_{E_z}(x, y) = \sum_{i \in \omega} \chi_{P^i_z}(x) \cdot \chi_{Q^i_z}(y)$.

After some "definable" refining repartitions using this uniformity and symmetrizations, we obtain uniformity for stable partitions:

Lemma

Suppose that $E \in \mathcal{B}_{X \times Y \times Z}$, $E_x \in \mathcal{B}_{Y \times Z}$ is μ-stable for almost all $x \in X$. Then there is a partition of $X \times Y$ into countably many sets $A^i \in \mathcal{B}_{X \times Y}, i \in \omega$, so that for almost every $x \in X$, $\left(A^i_x : i \in \omega\right)$ is a partition of Y into countably many sets perfect for E_x (viewed as a binary relation on $(X \times Y) \times Z$).

Partitioning $X \times Y$ into perfect sets

- Using this and some more work we obtain a partition of $X \times Y$ into perfect sets:
- **Proposition.** Suppose that $E \in \mathcal{B}_{X\times Y\times Z}$, $E_x \in \mathcal{B}_{Y\times Z}$ is μ-stable for almost all $x \in X$, and $E_y \in \mathcal{B}_{X\times Z}$ is μ-stable for almost all $y \in Y$. Then there is a partition of $X \times Y$ into $\mathcal{B}^E_{X\times Y}$-measurable sets perfect for E, viewed as a binary relation on $(X \times Y) \times Z$.
- However, we cannot hope to also partition Z into perfect sets for $E \subseteq (X \times Y) \times Z$, as we did with ordinary stability:
- Take $X = Y = Z = [0, 1]$ and let $E := \{(x, y, z) : x = y < z\}$, then E is slicewise stable. Place the Lebesgue measure on Z, and place discrete measures on X and Y which place a positive measure on each rational number in $[0, 1]$. Now if $A \subseteq Z$ has positive Lebesgue measure, we can always choose $q \in \mathbb{Q} \cap [0, 1]$ so that both $A \cap [0, q)$ and $A \cap (q, 1]$ have positive measure, that is $0 < \mu\left(E_{(q,q)} \cap A\right) < \mu(A)$. But $\mu(\{(q, q)\}) > 0$, so the set A is not perfect.

One direction of stability and the opposite slicewise stability

In this case the results we have suffice to give a positive answer to the question of Terry and Wolf.

Theorem

Assume that $E \in \mathcal{B}_{X \times Y \times Z}$ is μ-stable viewed as a binary relation between $X \times Y$ and Z, and the slices $E_z \in \mathcal{B}_{X \times Y}$ are μ-stable for almost all $z \in Z$. Then for every $\varepsilon > 0$ there exist finite partitions $X = \bigsqcup_{i \in I} X_i, Y = \bigsqcup_{j \in J} Y_j, Z = \bigsqcup_{k \in K} Z_k$ with $X_i \in \mathcal{B}_X, Y_j \in \mathcal{B}_Y, Z_k \in \mathcal{B}_Z$ so that for every $(i, j, k) \in I \times J \times K$ we have $\frac{\mu\left(E \cap \left(X_i \times Y_j \times Z_k\right)\right)}{\mu\left(X_i \times Y_j \times Z_k\right)} \in [0, \varepsilon) \cup (1 - \varepsilon, 1]$.

Partition into a combination of perfect sets and rectangles

But we only have slice-wise stability in all three directions! Some analysis of infinite (infinitely branching) trees of partitions, with infinite branches tackled by μ-stability on various repartitions of coordinates and slices, allows us to get:

Proposition. Suppose that $E \in \mathcal{B}_{X \times Y \times Z}$, the slices $E_x \in \mathcal{B}_{Y \times Z}$ are μ-stable for almost all $x \in X$, and the slices $E_y \in \mathcal{B}_{X \times Z}$ are μ-stable for almost all $y \in Y$. Then there exist a countable partition $X \times Y = \bigsqcup_{i \in \omega} A^i$ with each $A^i \in \mathcal{B}_{X \times Y}$ perfect for the relation $E \subseteq (X \times Y) \times Z$, and a countable partition $Y \times Z = \bigsqcup_{j \in \omega} B^j$ into rectangles $B^j = B^{j,Y} \times B^{j,Z}$ for some $B^{j,Y} \in \mathcal{B}_Y, B^{j,Z} \in \mathcal{B}_Z$, so that for each $i, j \in \omega$, either $A^i \wedge B^j \subseteq^0 E$ or $\left(A^j \wedge B^j\right) \cap E =^0 \emptyset$.

Finally...

- Finally, combining all of the above and some more repartitions, we obtain:
- **Proposition.** Suppose that $E \in \mathcal{B}_{X\times Y\times Z}$ is slicewise μ-stable. Then there exist a countable partition $X \times Y = \bigsqcup_{i\in\omega} A^i$ so that each A^i is perfect for the relation $E \subseteq (X \times Y) \times Z$, and $A^i = A^{i,X} \times A^{i,Y}$ is a rectangle with $A^{i,X} \in \mathcal{B}_X, A^{i,Y} \in \mathcal{B}_Y$.
- From which the main theorem quickly follows!
- A slicewise stable counterexample to stable hypergraph regularity: Let $X := \{0, 1, 2\}^\omega$, and $(x, y, z) \in E$ holds if, for the first n such that $|x(n), y(n), z(n)| > 1$, $|x(m), y(n), z(n)| = 3$. (At the first coordinate where they are not all the same, they are all different.)