

Mekler's construction and generalized stability

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Mekler's construction

- ▶ Let $p > 2$ be prime.
- ▶ Let T be any theory in a finite relational language.
- ▶ [Mekler'81] A uniform construction of a group $G(\mathcal{M})$ for every $\mathcal{M} \models T$, a theory T^* of all groups $\{G(\mathcal{M}) : \mathcal{M} \models T\}$ and an interpretation Γ of T in T^* s.t.:
 - ▶ T^* is a theory of nilpotent groups of class 2 and of exponent p ,
 - ▶ if $G \models T^*$, then $\exists \mathcal{M} \models T$ s.t. $G(\mathcal{M}) \cong G$,
 - ▶ For $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{M} \cong \mathcal{N} \iff G(\mathcal{M}) \cong G(\mathcal{N})$,
 - ▶ $\Gamma(G(\mathcal{M})) \cong \mathcal{M}$.
- ▶ Idea:
 - ▶ Bi-interpret \mathcal{M} with a *nice* graph C .
 - ▶ Define a group $G(C)$ generated freely by the vertices of C , imposing that two generators commute \iff they are connected by an edge in C .
- ▶ This kind of coding of graphs is known in probabilistic group theory, recursion theory, etc.

What model-theoretic properties are preserved?

- ▶ This is not a bi-interpretation (e.g., the resulting group is never ω -categorical), however some model-theoretic tameness properties are known to be preserved.
- ▶ [Mekler '81] For any cardinal κ , $\text{Th}(\mathcal{M})$ is κ -stable $\iff \text{Th}(G(\mathcal{M}))$ is κ -stable.
- ▶ [Baudisch, Pentzel '02] $\text{Th}(\mathcal{M})$ is simple $\iff \text{Th}(G(\mathcal{M}))$ is simple.
- ▶ [Baudisch '02] Assuming stability, $\text{Th}(\mathcal{M})$ is CM-trivial $\iff \text{Th}(G(\mathcal{M}))$ is CM-trivial.
- ▶ We investigate what further properties from Shelah's classification are preserved.

k -dependent theories

- ▶ We fix a complete theory T in a language \mathcal{L} . For $k \geq 1$ we define:

Definition

[Shelah]

- ▶ A formula $\phi(x; y_1, \dots, y_k)$ is k -dependent if there are no infinite sets $A_i = \{a_{i,j} : j \in \omega\} \subseteq M_{y_i}, i \in \{1, \dots, k\}$ in a model \mathcal{M} of T such that $A = \prod_{i=1}^k A_i$ is *shattered* by ϕ , where “ A shattered” means: for any $s \subseteq \omega^k$, there is some $b_s \in M_x$ s.t. $M \models \phi(b_s; a_{1,j_1}, \dots, a_{k,j_k}) \iff (j_1, \dots, j_k) \in s$.
- ▶ T is k -dependent if all formulas are k -dependent.
- ▶ T is *strictly* k -dependent if it is k -dependent, but not $(k - 1)$ -dependent.
- ▶ T is 1-dependent $\iff T$ is NIP.
- ▶ 1-dependent \subsetneq 2-dependent \subsetneq \dots as witnessed by e.g. the theory of the random k -hypergraph.

k -dependent fields?

- ▶ **Problem.** Are there strictly k -dependent fields, for $k > 1$?
- ▶ **Conjecture.** There are no *simple* strictly k -dependent fields, for $k > 1$.
- ▶ [Hempel '15] Let K be an infinite field.
 1. If $\text{Th}(K)$ is k -dependent, then K is Artin-Schreier closed.
 2. If K is a PAC field which is not separably closed, then $\text{Th}(K)$ is not k -dependent for any $k \in \omega$.
- ▶ (2) is due to Parigot for $k = 1$, and if K is pseudofinite, by Beyarslan K interprets the random k -hypergraph for all $k \in \omega$.

k -dependent groups

- ▶ Let T be a theory and G a type-definable group (over \emptyset), and $A \subseteq \mathbb{M}$ a small subset.
- ▶ Let G_A^{00} be the minimal type-definable over A subgroup of G of bounded index.

Fact

T is NIP $\implies G_A^{00} = G_{\emptyset}^{00}$ for all small A .

Example

Let $G := \bigoplus_{\omega} \mathbb{F}_p$. Let $\mathcal{M} := (G, \mathbb{F}_p, 0, +, \cdot)$ with \cdot the bilinear form $(a_i) \cdot (b_j) = \sum_i a_i b_i$ from G to \mathbb{F}_p .

Then G is 2-dependent and $G_A^{00} = \{g \in G : \bigcap_{a \in A} g \cdot a = 0\}$ — gets smaller when enlarging A .

Fact

[Shelah] Let T be 2-dependent. Then for a suitable cardinal κ , if $\mathcal{M} \prec \mathbb{M}$ is κ -saturated and $|B| < \kappa$, then $G_{M \cup B}^{00} = G_M^{00} \cap G_{A \cup B}^{00}$ for some $A \subseteq M$, $|A| < \kappa$.

- ▶ This can be viewed as a trace of modularity.

Mekler's construction preserves k -dependence

- ▶ No examples of strictly k -dependent groups for $k > 2$ were known.

Theorem

[C., Hempel '17] For any $k \in \omega$, $\text{Th}(\mathcal{M})$ k -dependent \iff
 $\text{Th}(G(\mathcal{M}))$ is k -dependent.

- ▶ Applying Mekler's construction to the random k -hypergraph, we get:

Corollary

For every $k \in \omega$, there is a strictly k -dependent pure group G_k (moreover, $\text{Th}(G_k)$ simple by Baudisch).

A proof for NIP, 1

- ▶ For a complete theory T , its *stability spectrum* is the function $f_T(\kappa) := \sup \{|S_1(M)| : M \models T, |M| = \kappa\}$.
- ▶ $\text{ded}(\kappa) := \sup \{|I| : I \text{ is a linear order with a dense subset of size } \kappa\}$.

Fact

[Shelah] Let the language of T be countable.

1. If T is NIP, then $f_T(\kappa) \leq (\text{ded } \kappa)^{\aleph_0}$ for all infinite cardinals κ .
 2. If T has IP, then $f_T(\kappa) = 2^\kappa$ for all infinite cardinals κ .
- ▶ Assuming GCH, $\text{ded } \kappa = 2^\kappa$ for all κ . On the other hand:
 - ▶ [Mitchell] For every cardinal κ with $\text{cf}(\kappa) > \aleph_0$, there is a forcing extension of the model of ZFC such that $(\text{ded } \kappa)^{\aleph_0} < 2^\kappa$.

A proof for NIP, 2

- ▶ The actual result in the original paper of Mekler is:

Fact

$f_{\text{Th}(G(\mathcal{M}))}(\kappa) \leq f_{\text{Th}(\mathcal{M})}(\kappa) + \aleph_0$ for all infinite cardinals κ .

- ▶ Hence if $\text{Th}(\mathcal{M})$ is NIP, then $f_{\text{Th}(G(\mathcal{M}))}(\kappa) \leq (\text{ded } \kappa)^{\aleph_0}$ for all κ , in all models of ZFC.
- ▶ Combining with Mitchell and using Schoenfield's absoluteness, $\text{Th}(G(\mathcal{M}))$ is NIP.
- ▶ Admittedly this is somewhat esoteric, and more importantly doesn't generalize to $k > 1$.

Characterization of k -dependence

- ▶ We want a formula-free characterization of k -dependence (in $\text{Th}(G(\mathcal{M}))$ we understand automorphisms, but not formulas).
- ▶ Let $\kappa := |T|^+$.

Fact

T is NIP \iff for every (\emptyset) -indiscernible sequence $(a_i : i \in \kappa)$ and b of finite tuples in \mathbb{M} , there is some $\alpha \in \kappa$ such that $(a_i : i > \alpha)$ is indiscernible over b .

- ▶ What is the analogue for k -dependence?

Generalized indiscernibles

- ▶ T is a theory in a language \mathcal{L} , $\mathbb{M} \models T$.

Definition

Let I be an \mathcal{L}_0 -structure. Say that $\bar{a} = (a_i : i \in I)$, with a_i a tuple in \mathbb{M} , is *I -indiscernible over $C \subseteq \mathbb{M}$* if for all i_1, \dots, i_n and j_1, \dots, j_n from I :

$$\text{qftp}_{\mathcal{L}_0}(i_1, \dots, i_n) = \text{qftp}_{\mathcal{L}_0}(j_1, \dots, j_n) \implies$$

$$\text{tp}_{\mathcal{L}}(a_{i_1}, \dots, a_{i_n}/C) = \text{tp}_{\mathcal{L}}(a_{j_1}, \dots, a_{j_n}/C).$$

- ▶ For \mathcal{L}_0 -structures I, J , say that $(b_j : j \in J)$ is *based on $(a_i : i \in I)$ over C* if for any finite set Δ of $\mathcal{L}(C)$ -formulas and any (j_0, \dots, j_n) from J there is some (i_1, \dots, i_n) from I s.t.
 $\text{qftp}_{\mathcal{L}_0}(j_1, \dots, j_n) = \text{qftp}_{\mathcal{L}_0}(i_1, \dots, i_n)$ and
 $\text{tp}_{\Delta}(b_{j_1}, \dots, b_{j_n}) = \text{tp}_{\Delta}(a_{i_1}, \dots, a_{i_n})$.
- ▶ We say that *I -indiscernibles exist* if for any \bar{a} indexed by I there is an I -indiscernible based on it.

Connection to structural Ramsey theory

- ▶ Implicitly used by Shelah already in the classification book, made explicit by Scow and others.

Definition

Let K be a class of finite \mathcal{L}_0 -structures. For $A, B \in K$, let $\binom{B}{A}$ be the set of all $A' \subseteq B$ s.t. $A' \cong A$.

K is *Ramsey* if for any $A, B \in K$ and $k \in \omega$ there is some $C \in K$ s.t. for any coloring $f : \binom{C}{A} \rightarrow k$, there is some $B' \in \binom{C}{B}$ s.t.

$f \upharpoonright \binom{B'}{A}$ is constant.

- ▶ Classical Ramsey theorem \iff the class of finite linear orders is Ramsey.

Fact

Let K be a Fraïssé class, and let I be its limit. If K is Ramsey, then I -indiscernibles exist.

Ordered random hypergraph indiscernibles

Fact

[Nes etril, R odl '77,'83] For any $k \in \omega$, the class of all finite ordered k -hypergraphs is Ramsey.

- ▶ Fix $k \in \omega$. Modifying their proof, we have existence of \mathcal{G} -indiscernibles for $\mathcal{G} = (P_1, \dots, P_k, R(x_1, \dots, x_k), <)$ the ordered k -partite random hypergraph (where $P_1 < \dots < P_k$).
- ▶ Let $\mathcal{O} = (P_1, \dots, P_k, <)$ denote the reduct of \mathcal{G} .
- ▶ Of course, $(a_g : g \in \mathcal{G})$ is \mathcal{O} -indiscernible / C implies it is \mathcal{G} -indiscernible / C .
- ▶ Clarifying Shelah,

Fact

[C., Palacin, Takeuchi '14] TFAE:

1. T is k -dependent.
2. For any $(a_g : g \in \mathcal{G})$ and b , with a_g, b finite tuples in \mathbb{M} , if $(a_g : g \in \mathcal{G})$ is \mathcal{G} -indiscernible over b and \mathcal{O} -indiscernible (over \emptyset), then it is \mathcal{O} -indiscernible over b .

Mekler's construction in more detail, 1

- ▶ A graph (binary, symmetric, irreflexive relation) C is *nice* if:
 - ▶ $\exists a \neq b$,
 - ▶ $\forall a \neq b \exists c (R(a, c) \wedge \neg R(b, c))$,
 - ▶ no triangles or squares.

Fact

Any structure in a finite relational language is bi-interpretable with a nice graph.

- ▶ Let $G \models \text{Th}(G(C))$, where $G(C)$ is generated freely by the vertices of C , and two generators commute \iff they are connected by an edge in C .
- ▶ We consider the following \emptyset -definable equivalence relations on G , each refining the previous one:
 - ▶ $g \sim h \iff C_G(g) = C_G(h)$,
 - ▶ $g \approx h \iff \exists r \in \omega, c \in Z(G) \text{ s.t. } g = h^r c$.
 - ▶ $g \equiv_Z h \iff gZ(G) = hZ(G)$.

Mekler's construction in more detail, 2

- ▶ $g \in G$ is of type q if \exists q -many \approx -classes in $[g]_{\sim}$.
- ▶ g is *isolated* if $[g]_{\sim} = [g]_{\equiv_Z}$.
- ▶ G can be partitioned into the following \emptyset -definable set:
 - ▶ non-isolated elements of type 1 — type 1^ν ,
 - ▶ isolated elements of type 1 — type 1^ι ,
 - ▶ elements of type p ,
 - ▶ elements of type $p - 1$.
- ▶ For every $g \in G$ of type p , the elements of G commuting with it are:
 - ▶ elements \sim -equivalent to g ,
 - ▶ an element b of type 1^ν together with the elements \sim -equivalent to b .
- ▶ Such a b is called a *handle* of g , and is definable from g up to \sim -equivalence.

Mekler's construction in more detail, 3

Definition

A set $X \subseteq G$ is a *transversal* if $X = X_{1^\nu} \sqcup X_p \sqcup X_\ell$, where:

1. X_{1^ν} : representatives for each \sim -class of elements of type 1^ν in G ;
2. X_p : representatives of \sim -classes of *proper* (i.e. not a product of any elements of type 1^ν) elements of type p , maximal with the property that if $Y \subseteq X_p$ is a finite set of elements with the same handle, then Y is independent modulo the subgroup generated by all elements of type 1^ν and $Z(G)$;
3. X_ℓ : representatives of \sim -classes of proper elements of type 1^ℓ , maximal independent modulo the subgroup generated by all elements of types 1^ν and p in G , together with $Z(G)$.

Mekler's construction in more detail, 4

- ▶ $C = (V, R)$ is interpreted in G as $\Gamma(G)$:
 - ▶ $V = \{g \in G : g \text{ is of type } 1^\nu, g \notin Z(G)\} / \approx,$
 - ▶ $([g]_{\approx}, [h]_{\approx}) \in R \iff g, h \text{ commute.}$
- ▶ For X a transversal of G , $\Gamma(X_\nu)$ is isomorphic to C .
- ▶ Let $G \models \text{Th}(G(C))$ and X a transversal of G . There is a subgroup (elementary abelian p -group) H of $Z(G)$ s.t.
 $G \cong \langle X \rangle \times H$.
- ▶ There is some canonicity about this choice: $\langle X \rangle' = G'$ for any transversal X of G .

Mekler's construction in more detail, summarizing

- ▶ For any partial transversal X' and any linearly independent over G' subset H' of $Z(G)$, we can find a transversal $X \supseteq X'$ and a maximal set $H \supseteq H'$ s.t. $G = \langle X \rangle \times \langle H \rangle$.
- ▶ **Lemma.** Both conditions on X' and H' are type-definable.
- ▶ If $Y, Z \subseteq X$ and $h : Y \rightarrow Z$ is a bijection respecting the 1^ν -, p -, and 1^ι -parts and the handles, and $\text{tp}_\Gamma(Y_\nu) = \text{tp}_\Gamma(h(Y_\nu))$, then $\text{tp}_G(Y) = \text{tp}_G(h(Y))$.
- ▶ Moreover, assuming saturation, h extends to an automorphism of G by gluing it with any automorphism of $\langle H \rangle$.

Sketch of the proof, 1

- ▶ Let $G \models \text{Th}(G(\mathcal{M}))$ be a monster model, and $\phi(x; y_1, \dots, y_k)$ not k -dependent.
- ▶ Choose a transversal X and $H \subseteq Z(G)$ s.t. $G = \langle X \rangle \times \langle H \rangle$.
- ▶ Compactness: a very large witness $(a_g : g \in \mathcal{G})$ to the failure of k -dependence, shattered by ϕ .
- ▶ For cardinality reasons, may assume $a_g = t(\bar{x}_g, \bar{h}_g)$ for some \mathcal{L}_G -term t and \bar{x}_g from X and \bar{h}_g from H .
- ▶ Can close under handles and, changing the formula, replace the original shattered set by $(\bar{x}_g \bar{h}_g : g \in \mathcal{G})$.
- ▶ Using type-definability of partial transversals, etc. and existence of \mathcal{G} -indiscernibles, can assume $(\bar{x}_g \bar{h}_g : g \in \mathcal{G})$ is \mathcal{O} -indiscernible (possibly changing the transversal to some X', H').
- ▶ As $(\bar{x}_g \bar{h}_g : g \in \mathcal{G})$ is shattered, can choose $b = s(\bar{y}, \bar{k}) \in G$ with $\bar{y} \in X', \bar{k} \in H'$ s.t. $\phi(b; y_1, \dots, y_k)$ cuts out *exactly* the edge relation of the random k -hypergraph \mathcal{G} .

Sketch of the proof, 2

- ▶ Using existence of \mathcal{G} -indiscernibles again, can assume that $(\bar{x}_g \bar{h}_g : g \in \mathcal{G})$ is \mathcal{G} -indiscernible over b (needs some argument, replacing X', H' by some X'', H'').
- ▶ Using that $\text{Th}(\langle X \rangle)$ and $\text{Th}(\langle H \rangle)$ are k -dependent by assumption (hence \mathcal{G} -indiscernibility collapses to \mathcal{O} -indiscernibility in them by the characterization above), can build an automorphism of G (glueing separate automorphisms of $\langle X'' \rangle$ and $\langle H'' \rangle$ together by the lemma above) σ such that:
- ▶ for some finite tuples of indices \bar{g}, \bar{h} of the same type in \mathcal{O} , but **not** in \mathcal{G} , σ fixes b and sends $(\bar{x}_g \bar{h}_g : g \in \bar{g})$ to $(\bar{x}_h \bar{h}_h : h \in \bar{h})$.
- ▶ — contradiction to the choice of b .

Other results and directions

Theorem

[C., Hempel '17] $\text{Th}(\mathcal{M})$ is $\text{NTP}_2 \iff \text{Th}(G(\mathcal{M}))$ is NTP_2 .

► **Problem.**

- Are there pseudofinite strictly k -dependent groups?
- Are there ω -categorical strictly k -dependent groups?