The Viscosity Formulation in [Ram et al., 2015] Missed a Δt

Chenfanfu Jiang

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In [Ram et al., 2015], by assuming backward Euler time integration ($\mathbf{x} = \mathbf{x}^n + \mathbf{v}\Delta t$), the Newtonian viscosity is reformulated as a discrete position-dependent potential energy (Eq. 10):

$$\Phi(\mathbf{x}) = \sum_p V_p^n \mu \epsilon_p : \epsilon_p, \tag{1}$$

where

$$\epsilon_p = \frac{1}{2}\left(\nabla \mathbf{v} + \nabla \mathbf{v}^T\right), \tag{2}$$

$$\mathbf{v} = \frac{1}{\Delta t}(\mathbf{x} - \mathbf{x}^n). \tag{3}$$

Force per MPM grid node is then

$$\mathbf{f}_i = -\frac{\partial \Phi}{\partial \mathbf{x}_i} = -\sum_p V_p^n \frac{\mu}{\Delta t}(\nabla \mathbf{v} + \nabla \mathbf{v}^T)\nabla w_{ip}^n \tag{4}$$

Claim: This energy is wrong and misses a Δt scaling. The correct energy should be

$$\Phi(\mathbf{x})^{\text{correct}} = \sum_p V_p^n \Delta t \mu \epsilon_p : \epsilon_p. \tag{5}$$

Proof 1: by unit analysis.

- Φ $J = kg \cdot m^2 \cdot s^{-2}$
- V_p^n m^3
- μ $N \cdot s \cdot m^{-2} = Pa \cdot s = kg \cdot m^{-1} \cdot s^{-1}$
- ϵ s^{-1}
- $\epsilon : \epsilon$ s^{-2}

Apparently Eq. (1) fails. The left is $kg \cdot m^2 \cdot s^{-2}$, but the right is $m^3 \cdot kg \cdot m^{-1} \cdot s^{-1} \cdot s^{-2} = kg \cdot m^2 \cdot s^{-3}$. Apparently the right misses a scaling with unit s.

Proof 2: by comparing to the weak form. Newtonian viscosity corresponds to Cauchy stress

$$\sigma = 2\mu\epsilon = \mu\left(\nabla \mathbf{v} + \nabla \mathbf{v}^T\right), \tag{6}$$

where μ is dynamics viscosity. Nodal force derived from the Galerkin weak form is

$$\mathbf{f}_i = -\int_{\Omega^n} \sigma \nabla N_i^n(\mathbf{x}) d\mathbf{x}. \tag{7}$$

Using particle quadratures, it becomes

$$\mathbf{f}_i = -\sum_p V_p^n \sigma_p \nabla w_{ip}^n = -\sum_p V_p^n \mu(\nabla \mathbf{v} + \nabla \mathbf{v}^T)\nabla w_{ip}^n. \tag{8}$$

Comparing it to Eq. (4), we can clearly see that Eq. (4) misses a Δt scaling factor.

Proof 3: the wrong force grows unboundly under Δt refinement.

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References

[Ram et al., 2015] Ram, D., Gast, T., Jiang, C., Schroeder, C., Stomakhin, A., Teran, J., and Kavehpour, P. (2015). A material point method for viscoelastic fluids, foams and sponges. In *Proc ACM SIGGRAPH/Eurograph Symp Comp Anim*, pages 157–163.