Anisotropic elastoplasticity for cloth, knit and hair frictional contact supplementary technical document

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1 QR differentiation

Here we discuss how to compute $\delta\mathbf{Q}$ and $\delta\mathbf{R}$ from $\mathbf{F}$ and $\delta\mathbf{F}$. This is used for computing the linearized force in implicit time integration.
If $\mathbf{F} = \mathbf{QR}$ where $\mathbf{Q}$ is orthonormal with $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$, and

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ & r_{22} & r_{23} \\ & & r_{33} \end{bmatrix}$$

is upper triangular, and we define the function $\Psi(\mathbf{F}) = \hat{\Psi}(\mathbf{R})$ then we have

$$\begin{aligned} \delta\mathbf{F} &= \mathbf{Q}\delta\mathbf{R} + \delta\mathbf{Q}\mathbf{R} \\ \delta\Psi(\mathbf{F}) &= \delta\hat{\Psi}(\mathbf{R}) \\ \delta\mathbf{Q}^T\mathbf{Q} + \mathbf{Q}^T\delta\mathbf{Q} &= \mathbf{0} \\ \delta\mathbf{R} & \text{ is upper triangular} \end{aligned}$$

$$\begin{aligned} \mathbf{Q}^T\delta\mathbf{F} &= \delta\mathbf{R} + \mathbf{Q}^T\delta\mathbf{Q}\mathbf{R} \\ \mathbf{\Omega} = \mathbf{Q}^T\delta\mathbf{Q} &= \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \\ \mathbf{Q}^T\delta\mathbf{F} &= \delta\mathbf{R} + \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}\mathbf{R} \\ \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} &= \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} + \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix} \end{aligned}$$

This implies

$$\begin{aligned} d &= \omega_3 r_{11} \\ g &= -\omega_2 r_{11} \\ h &= -\omega_2 r_{12} + \omega_1 r_{22} \end{aligned}$$

We can use these three equations to solve for $\omega_1, \omega_2, \omega_3$. After that, we can construct $\delta\mathbf{Q} = \mathbf{Q}\Omega$ and then $\delta\mathbf{R} = \mathbf{Q}^T\delta\mathbf{F} - \mathbf{Q}^T\delta\mathbf{Q}\mathbf{R}$.

The corresponding 2D result is

$$a = -\frac{(\mathbf{Q}^T \delta\mathbf{F})_{21}}{r_{11}}$$
$$\delta\mathbf{Q} = a\mathbf{Q}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$\delta\mathbf{R} = \mathbf{Q}^T\delta\mathbf{F} - \mathbf{Q}^T\delta\mathbf{Q}\mathbf{R}$$

2 Computing stress

$$\begin{aligned}
\delta\Psi &= \delta\hat{\Psi} \\
\frac{\partial\Psi}{\partial\mathbf{F}} : \delta\mathbf{F} &= \frac{\partial\hat{\Psi}}{\partial\mathbf{R}} : \delta\mathbf{R} \\
\frac{\partial\Psi}{\partial\mathbf{F}} : (\mathbf{Q}\delta\mathbf{R} + \delta\mathbf{Q}\mathbf{R}) &= \frac{\partial\hat{\Psi}}{\partial\mathbf{R}} : \delta\mathbf{R} \\
\frac{\partial\Psi}{\partial\mathbf{F}} : (\mathbf{Q}\delta\mathbf{R}) + \frac{\partial\Psi}{\partial\mathbf{F}} : (\delta\mathbf{Q}\mathbf{R}) &= \frac{\partial\hat{\Psi}}{\partial\mathbf{R}} : \delta\mathbf{R} \qquad (1)
\end{aligned}$$

It can be shown that the above holds for any $\delta\mathbf{Q}$ and $\delta\mathbf{R}$ that satisfy $\delta\mathbf{Q}^T\mathbf{Q} + \mathbf{Q}^T\delta\mathbf{Q} = \mathbf{0}$ and $\delta\mathbf{R}$ being upper triangular. Specifically, it can be shown that given arbitrary $\delta\mathbf{Q}, \delta\mathbf{R}$ with $\delta\mathbf{Q}^T\mathbf{Q} + \mathbf{Q}^T\delta\mathbf{Q} = \mathbf{0}$ and $\delta\mathbf{R}$ upper triangular, $\exists\delta\mathbf{F}$ such that

$$\delta\mathbf{R} = \frac{\partial\mathbf{R}}{\partial\mathbf{F}}(\mathbf{F}) : \delta\mathbf{F}, \text{ and } \delta\mathbf{Q} = \frac{\partial\mathbf{Q}}{\partial\mathbf{F}}(\mathbf{F}) : \delta\mathbf{F}.$$

Using $\text{null}[\frac{\partial\mathbf{Q}}{\partial\mathbf{F}}(\mathbf{F})] = \text{span}\{\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, \mathbf{N}_4, \mathbf{N}_5, \mathbf{N}_6\}$, for all upper triangular $\delta\mathbf{R}$, $\exists\delta\mathbf{F}_Q \in \text{null}[\frac{\partial\mathbf{Q}}{\partial\mathbf{F}}(\mathbf{F})]^\perp$ such that $\delta\mathbf{Q} = \frac{\partial\mathbf{Q}}{\partial\mathbf{F}}(\mathbf{F}) : \delta\mathbf{F}_Q = \frac{\partial\mathbf{Q}}{\partial\mathbf{F}}(\mathbf{F}) : \delta(\mathbf{F}_Q + \mathbf{F}_R)\ \forall\mathbf{F}_R \in \text{null}[\frac{\partial\mathbf{Q}}{\partial\mathbf{F}}(\mathbf{F})]$. Further, it can be shown that $\mathbf{Q}\delta\mathbf{R} + \delta\mathbf{Q}\mathbf{R} - \delta\mathbf{F}_Q \in \text{null}[\frac{\partial\mathbf{Q}}{\partial\mathbf{F}}(\mathbf{F})]$ and with $\delta\mathbf{F}_R = \mathbf{Q}\delta\mathbf{R} + \delta\mathbf{Q}\mathbf{R} - \delta\mathbf{F}_Q$, $\delta\mathbf{F} = \delta\mathbf{F}_Q + \delta\mathbf{F}_R$ produces

$$\delta\mathbf{R} = \frac{\partial\mathbf{R}}{\partial\mathbf{F}}(\mathbf{F}) : \delta\mathbf{F}, \text{ and } \delta\mathbf{Q} = \frac{\partial\mathbf{Q}}{\partial\mathbf{F}}(\mathbf{F}) : \delta\mathbf{F}.$$

If we choose $\delta\mathbf{Q} = \mathbf{0}$ in Equation (1), then

$$\begin{aligned}
\frac{\partial\Psi}{\partial\mathbf{F}} : (\mathbf{Q}\delta\mathbf{R}) &= \frac{\partial\hat{\Psi}}{\partial\mathbf{R}} : \delta\mathbf{R} \\
(\mathbf{Q}^T\frac{\partial\Psi}{\partial\mathbf{F}}) : \delta\mathbf{R} &= \frac{\partial\hat{\Psi}}{\partial\mathbf{R}} : \delta\mathbf{R}
\end{aligned}$$

Recall $\delta\mathbf{R}$ is any upper triangular matrix, therefore $(\mathbf{Q}^T\frac{\partial\Psi}{\partial\mathbf{F}})$ and $\frac{\partial\hat{\Psi}}{\partial\mathbf{R}}$ have the same upper triangular part. Further more, since $\mathbf{R}^T$ is lower triangular, it is easy to show (entry wise provable) that $\mathbf{Q}^T\frac{\partial\Psi}{\partial\mathbf{F}}\mathbf{R}^T$ and $\frac{\partial\hat{\Psi}}{\partial\mathbf{R}}\mathbf{R}^T$ have the same upper triangular part.

If we choose $\delta\mathbf{R} = \mathbf{0}$ in Equation (1), then

$$\frac{\partial\Psi}{\partial\mathbf{F}} : (\delta\mathbf{Q}\mathbf{R}) = \mathbf{0}$$
$$(\frac{\partial\Psi}{\partial\mathbf{F}}\mathbf{R}^T) : \delta\mathbf{Q} = \mathbf{0}$$
$$(\frac{\partial\Psi}{\partial\mathbf{F}}\mathbf{R}^T) : (\delta\mathbf{Q}\mathbf{Q}^T\mathbf{Q}) = \mathbf{0}$$
$$(\frac{\partial\Psi}{\partial\mathbf{F}}\mathbf{R}^T\mathbf{Q}^T) : (\delta\mathbf{Q}\mathbf{Q}^T) = \mathbf{0}$$
$$(\frac{\partial\Psi}{\partial\mathbf{F}}\mathbf{F}^T) : (\delta\mathbf{Q}\mathbf{Q}^T) = \mathbf{0}$$

Since $\delta\mathbf{Q}\mathbf{Q}^T$ is an arbitrary skew symmetric matrix (due to $\delta\mathbf{Q}^T\mathbf{Q} + \mathbf{Q}^T\delta\mathbf{Q} = \mathbf{0}$), for the above equation to hold, we know $\frac{\partial\Psi}{\partial\mathbf{F}}\mathbf{F}^T$ has to be symmetric. This also proves that Kirchoff stress $\tau = \frac{\partial\Psi}{\partial\mathbf{F}}\mathbf{F}^T$ is symmetric without needing to use conservation of angular momentum. Now we have

$$\tau = \tau^T$$
$$\frac{\partial\Psi}{\partial\mathbf{F}}\mathbf{F}^T = \mathbf{F}(\frac{\partial\Psi}{\partial\mathbf{F}})^T$$
$$\frac{\partial\Psi}{\partial\mathbf{F}}\mathbf{R}^T\mathbf{Q}^T = \mathbf{Q}\mathbf{R}(\frac{\partial\Psi}{\partial\mathbf{F}})^T$$
$$\mathbf{Q}^T\frac{\partial\Psi}{\partial\mathbf{F}}\mathbf{R}^T = \mathbf{R}(\frac{\partial\Psi}{\partial\mathbf{F}})^T\mathbf{Q}$$

i.e., $\mathbf{Q}^T\frac{\partial\Psi}{\partial\mathbf{F}}\mathbf{R}^T$ is symmetric.
In summary, $\mathbf{Q}^T\frac{\partial\Psi}{\partial\mathbf{F}}\mathbf{R}^T$ and $\frac{\partial\hat{\Psi}}{\partial\mathbf{R}}\mathbf{R}^T$ have the same upper triangular part. $\mathbf{Q}^T\frac{\partial\Psi}{\partial\mathbf{F}}\mathbf{R}^T$ is symmetric. We further denote this tensor with $\mathbf{A} := \mathbf{Q}^T\frac{\partial\Psi}{\partial\mathbf{F}}\mathbf{R}^T = \mathbf{Q}^T\frac{\partial\Psi}{\partial\mathbf{F}}\mathbf{F}^T\mathbf{Q} = \mathbf{Q}^T\tau\mathbf{Q}$. Therefore $\mathbf{A}$ is just τ written in the $\mathbf{Q}$ basis.

3 A curve in 3D

We can construct the upper triangular part of $\mathbf{A}$ with $\frac{\partial\hat{\Psi}}{\partial\mathbf{R}}\mathbf{R}^T$, then fill the rest using symmetry of $\mathbf{A}$. Assuming

$$\hat{\Psi} = f(r_{11}) + \frac{1}{2}g(r_{12}^2 + r_{13}^2) + h(r_{22}, r_{23}, r_{33}),$$

then

$$\frac{\partial\hat{\Psi}}{\partial\mathbf{R}} = \begin{bmatrix} f' & g'\mathbf{r}^T \\ \mathbf{0} & \hat{\mathbf{P}} \end{bmatrix},$$

where $\mathbf{r} = (r_{12}, r_{13})^T$, $\mathbf{R} = \begin{bmatrix} r_{11} & \mathbf{r}^T \\ \mathbf{0} & \hat{\mathbf{R}} \end{bmatrix}$,$\hat{\mathbf{P}} = \frac{\partial h}{\partial\hat{\mathbf{R}}}$.
We can show

$$\mathbf{A} = \begin{bmatrix} f'r_{11} + g'\mathbf{r}^T\mathbf{r} & g'\mathbf{r}^T\hat{\mathbf{R}}^T \\ g'\mathbf{r}^T\hat{\mathbf{R}}^T & \hat{\mathbf{P}}\hat{\mathbf{R}}^T \end{bmatrix}$$

Here, we choose $\hat{\mathbf{P}}\hat{\mathbf{R}}^T$ to be the dilational part of the Kirchoff stress from the 2×2 Stvk Hencky Drucker-Prager. i.e., assuming we get some $\hat{\tau}$ after the return mapping of dry sand from an input $\mathbf{F} = \hat{\mathbf{R}}$, we replace the bottom right corner of $\mathbf{A}$ with $p\mathbf{I}$, where $p = tr(\hat{\tau})/2$.

Under such a choice,

$$\mathbf{A} = \begin{bmatrix} f'r_{11} + g'\mathbf{r}^T\mathbf{r} & g'\mathbf{r}^T\hat{\mathbf{R}}^T \\ g'\mathbf{r}^T\hat{\mathbf{R}}^T & p\mathbf{I} \end{bmatrix}.$$

Mohr friction criteria leads to the following maximization problem:
Maximize $\mathbf{d}^T\tau\mathbf{n} + c_F\mathbf{n}^T\tau\mathbf{n}$ over all possible $\mathbf{n}$ that is perpendicular to the fiber direction, with all $\mathbf{d}$ perpendicular to $\mathbf{n}$ and has unit length.
i.e.,
Maximize $\mathbf{d}^T\mathbf{QAQ}^T\mathbf{n} + c_F\mathbf{n}^T\mathbf{QAQ}^T\mathbf{n}$ over all possible $\mathbf{n}$ that is perpendicular to the fiber direction, with all $\mathbf{d}$ perpendicular to $\mathbf{n}$ and has unit length.
i.e.,
Maximize $\tilde{\mathbf{d}}^T\mathbf{A}\tilde{\mathbf{n}} + c_F\tilde{\mathbf{n}}^T\mathbf{A}\tilde{\mathbf{n}}$ over all possible $\mathbf{n}$ that is perpendicular to the fiber direction, with all $\mathbf{d}$ perpendicular to $\mathbf{n}$ and has unit length, where $\tilde{\mathbf{n}} = \mathbf{Q}^T\mathbf{n}$ and $\tilde{\mathbf{d}} = \mathbf{Q}^T\mathbf{d}$. Since in our discretization, the fiber direction is $\mathbf{q}_1$, therefore $\mathbf{q}_1^T\mathbf{n} = 0$, then we know $\tilde{\mathbf{n}} = (0, c, s)$ for some θ.
i.e.,
Maximize $\tilde{\mathbf{d}}^T\mathbf{A}\tilde{\mathbf{n}} + c_F\tilde{\mathbf{n}}^T\mathbf{A}\tilde{\mathbf{n}}$ over all possible $\mathbf{n}$ that is $(0, c, s)$ over all θ, with all $\tilde{\mathbf{d}}$ perpendicular to $\tilde{\mathbf{n}}$ and has unit length.
Lagrangian multiplier can solve this problem and gives the maximum $\left\|\hat{\mathbf{R}}\mathbf{r}\right\| g' + c_F p$. If we choose $g(x) = x$, then we have our yield surface $\left\|\hat{\mathbf{R}}\mathbf{r}\right\| + c_F p < 0$. The return mapping is then simply a scaling on $\mathbf{r}$ so that the yield criteria is satisfied.

4 A surface in 3D

Surface is very similar to curve in our framework. With our codimensional discretization, we map the triangle back to x-y plane. If the input 3D triangle at rest is $\hat{\mathbf{A}}, \hat{\mathbf{B}}, \hat{\mathbf{C}}$, we can define $\hat{\mathbf{D}}_1 = \hat{\mathbf{B}} - \hat{\mathbf{A}}$ and $\hat{\mathbf{D}}_2 = \hat{\mathbf{C}} - \hat{\mathbf{A}}$. This forms an imaginary $\hat{\mathbf{D}}_s$, whose QR decomposition gives us the rotation from x-y plane of this triangle $\hat{\mathbf{Q}}$, as well as the top part of $\hat{\mathbf{R}}_{3\times 2}$ being the 2×2 version $\mathbf{D}_m$. The third column of $\hat{\mathbf{Q}}$ is the rotated $\mathbf{D}_3$ where $\mathbf{D}_3 = \mathbf{e}_3$.
With these precomputations, for any triangle $\mathbf{d}_1, \mathbf{d}_2$ in world space, we can construct the full $\mathbf{F}$ as

$$\mathbf{F} = [\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3] \begin{bmatrix} \mathbf{D}_m^{-1} & \\ & 1 \end{bmatrix}.$$

In MPM, we have $\mathbf{d}_3^{n+1} = (\mathbf{I} + \Delta t \nabla \mathbf{v})\mathbf{d}_3^n$, with $\mathbf{d}_3^0 = \hat{\mathbf{q}}_3$. Keep in mind that $\mathbf{D}_m$ is upper triangular, therefore $\mathbf{D}_m^{-1}$ is upper triangular.
Now do the thin QR decomposition

$$[\mathbf{d}_1, \mathbf{d}_2] = [\mathbf{q}_1, \mathbf{q}_2]\tilde{\mathbf{R}}$$

where $\tilde{\mathbf{R}}$ is 2×2 upper.
If we construct $\mathbf{q}_3 = \mathbf{q}_1 \times \mathbf{q}_2$, then

$$[\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3] = [\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3] \begin{bmatrix} \tilde{\mathbf{R}} & (h_x, h_y)^T \\ \mathbf{0}^T & h_z \end{bmatrix}$$

where $\mathbf{Qh} := \mathbf{d}_3$. Now we have constructed the (unique) QR decomposition of $\mathbf{F}$ as

$$\mathbf{F} = [\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3] \begin{bmatrix} \tilde{\mathbf{R}} & (h_x, h_y)^T \\ \mathbf{0}^T & h_z \end{bmatrix} \begin{bmatrix} \mathbf{D}_m^{-1} & \\ & 1 \end{bmatrix} := \mathbf{QR} = \mathbf{Q} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ & r_{22} & r_{23} \\ & & r_{33} \end{bmatrix}$$

We can see $\mathbf{h} = \mathbf{r}_3 = \mathbf{Q}^T\mathbf{d}_3$.

The previous lemma in curve still holds: $\mathbf{Q}^T \frac{\partial \Psi}{\partial \mathbf{F}} \mathbf{R}^T$ and $\frac{\partial \hat{\Psi}}{\partial \mathbf{R}} \mathbf{R}^T$ have the same upper triangular part. $\mathbf{Q}^T \frac{\partial \Psi}{\partial \mathbf{F}} \mathbf{R}^T$ is symmetric. We further denote this tensor with $\mathbf{A} := \mathbf{Q}^T \frac{\partial \Psi}{\partial \mathbf{F}} \mathbf{R}^T = \mathbf{Q}^T \frac{\partial \Psi}{\partial \mathbf{F}} \mathbf{F}^T \mathbf{Q} = \mathbf{Q}^T \tau \mathbf{Q}$. Therefore $\mathbf{A}$ is just τ written in the $\mathbf{Q}$ basis. We can construct the upper triangular part of $\mathbf{A}$ with $\frac{\partial \hat{\Psi}}{\partial \mathbf{R}} \mathbf{R}^T$, then fill the rest using symmetry of $\mathbf{A}$.

4.1 Surface elastoplasticity

Recall $\mathbf{D}_3 = \mathbf{e}_3$,

$$\mathbf{F} = \mathbf{Q} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ & r_{22} & r_{23} \\ & & r_{33} \end{bmatrix}$$

Physically, the top left is in plane (x-y) deformation of the triangle, since

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ & r_{22} & r_{23} \\ & & r_{33} \end{bmatrix} \mathbf{e}_3 = \mathbf{r}_3,$$

we know $\mathbf{r}_3$ represents the deformation of $\mathbf{D}_3$. From $\mathbf{h} = \mathbf{r}_3 = \mathbf{Q}^T \mathbf{d}_3$, we can say $|\mathbf{r}_3|$ is the length change, $r_{13}^2 + r_{23}^2$ is the shearing (or the deviation from being perpendicular to the x-y triangle plane). Note that shearing also pernalizes length change.
From these, we can define

$$\hat{\Psi} = f(r_{33}) + \frac{1}{2} g(r_{13}^2 + r_{23}^2) + h(r_{11}, r_{12}, r_{21}, r_{22}),$$

then

$$\frac{\partial \hat{\Psi}}{\partial \mathbf{R}} = \begin{bmatrix} \hat{\mathbf{P}} & g' \mathbf{r} \\ \mathbf{0}^T & f' \end{bmatrix},$$

where $\mathbf{r} = (r_{13}, r_{23})^T$, $\mathbf{R} = \begin{bmatrix} \hat{\mathbf{R}} & \mathbf{r} \\ \mathbf{0}^T & r_{33} \end{bmatrix}$,$\hat{\mathbf{P}} = \frac{\partial h}{\partial \hat{\mathbf{R}}}$.
We can show

$$\mathbf{A} = \begin{bmatrix} \hat{\mathbf{P}} \hat{\mathbf{R}}^T + g' \mathbf{r} \mathbf{r}^T & g' r_{33} \mathbf{r} \\ g' r_{33} \mathbf{r}^T & f' r_{33} \end{bmatrix}$$

Mohr friction criteria leads to the following maximization problem:
Maximize $\mathbf{d}^T \tau \mathbf{n} + c_F \mathbf{n}^T \tau \mathbf{n}$ over all possible $\mathbf{n}$ that is perpendicular to the manifold plane, with all $\mathbf{d}$ perpendicular to $\mathbf{n}$ and has unit length.
i.e.,
Maximize $\mathbf{d}^T \mathbf{Q} \mathbf{A} \mathbf{Q}^T \mathbf{n} + c_F \mathbf{n}^T \mathbf{Q} \mathbf{A} \mathbf{Q}^T \mathbf{n}$ over all possible $\mathbf{n}$ that is perpendicular to the manifold plane, with all $\mathbf{d}$ perpendicular to $\mathbf{n}$ and has unit length.
i.e.,
Maximize $\tilde{\mathbf{d}}^T \mathbf{A} \tilde{\mathbf{n}} + c_F \tilde{\mathbf{n}}^T \mathbf{A} \tilde{\mathbf{n}}$ over all possible $\mathbf{n}$ that is perpendicular to the manifold plane, with all $\mathbf{d}$ perpendicular to $\mathbf{n}$ and has unit length, where $\tilde{\mathbf{n}} = \mathbf{Q}^T \mathbf{n}$ and $\tilde{\mathbf{d}} = \mathbf{Q}^T \mathbf{d}$.
$\mathbf{n}$ perpendicular to manifold plane means $\mathbf{n} = k(\mathbf{d}_1 \times \mathbf{d}_2)$ for some k (unit length constraint is extra). Recall

$$[\mathbf{d}_1, \mathbf{d}_2] = [\mathbf{q}_1, \mathbf{q}_2] \tilde{\mathbf{R}},$$

where $\tilde{\mathbf{R}}$ is upper triangular 2×2, this means $k(\mathbf{d}_1 \times \mathbf{d}_2) = z(\mathbf{q}_1 \times \mathbf{q}_2)$ for some z. Therefore, $\mathbf{n} = \pm \mathbf{q}_3$. Therefore $\tilde{\mathbf{n}} = \mathbf{Q}^T \mathbf{n} = \pm \mathbf{e}_3$.
i.e.,
Maximize $\tilde{\mathbf{d}}^T \mathbf{A} \tilde{\mathbf{n}} + c_F \tilde{\mathbf{n}}^T \mathbf{A} \tilde{\mathbf{n}}$ over $\tilde{\mathbf{n}} = \pm \mathbf{e}_3$, with all $\tilde{\mathbf{d}} = (c, s, 0)$ for some θ

The maximum is

$$\pm g' r_{33}|\mathbf{r}| + c_F f' r_{33}$$

Assume $f(x) = \frac{1}{3}k(1-x)^3$ for $x \leq 1$, 0 otherwise. $g(x) = \gamma x$. When $r_{33} > 1$, $f = 0$, the maximum is

$$\gamma r_{33}|\mathbf{r}|$$

. In this case the return mapping is making $\mathbf{r}$ to be zero.
When $r_{33} < 1$, the maximum is

$$\pm \gamma r_{33}|\mathbf{r}| - c_F k(r_{33}-1)^2 r_{33}$$

. The yield surface is therefore

$$max(\pm \frac{\gamma}{k} r_{33}|\mathbf{r}| - c_F(r_{33}-1)^2 r_{33}) \leq 0$$

If r_{33} is negative (corresponding to inverted collision), the max should choose $-\frac{\gamma}{k}$. The return mapping is setting $\mathbf{r}$ to be $\mathbf{0}$.
Otherwise, try to scale $\mathbf{r}$ (when necessary) to satisfy

$$\frac{\gamma}{k}|\mathbf{r}| - c_F(r_{33}-1)^2 \leq 0$$

5 A curve in 2D

2D derivation follows 3D curve derivation: $\mathbf{Q}^T \frac{\partial \Psi}{\partial \mathbf{F}} \mathbf{R}^T$ and $\frac{\partial \hat{\Psi}}{\partial \mathbf{R}} \mathbf{R}^T$ have the same upper triangular part. $\mathbf{Q}^T \frac{\partial \Psi}{\partial \mathbf{F}} \mathbf{R}^T$ is symmetric. We further denote this tensor with $\mathbf{A} := \mathbf{Q}^T \frac{\partial \Psi}{\partial \mathbf{F}} \mathbf{R}^T = \mathbf{Q}^T \frac{\partial \Psi}{\partial \mathbf{F}} \mathbf{F}^T \mathbf{Q} = \mathbf{Q}^T \tau \mathbf{Q}$. Therefore $\mathbf{A}$ is just τ written in the $\mathbf{Q}$ basis.
The energy choice is

$$\hat{\Psi} = f(r_{11}) + \frac{1}{2} g(r_{12}^2) + h(r_{22}),$$

then

$$\frac{\partial \hat{\Psi}}{\partial \mathbf{R}} = \begin{bmatrix} f' & g' r_{12} \\ 0 & h' \end{bmatrix},$$

where $\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}$,$\hat{\mathbf{P}} = \frac{\partial h}{\partial \hat{\mathbf{R}}}$.
We can show

$$\mathbf{A} = \begin{bmatrix} f' r_{11} + g' r_{12}^2 & g' r_{12} r_{22} \\ g' r_{12} r_{22} & h' r_{22} \end{bmatrix}$$

Mohr friction criteria leads to the following maximization problem:
Maximize $\mathbf{d}^T \tau \mathbf{n} + c_F \mathbf{n}^T \tau \mathbf{n}$ over all possible $\mathbf{n}$ that is perpendicular to the fiber direction, with all $\mathbf{d}$ perpendicular to $\mathbf{n}$ and has unit length.
i.e.,
Maximize $\mathbf{d}^T \mathbf{Q} \mathbf{A} \mathbf{Q}^T \mathbf{n} + c_F \mathbf{n}^T \mathbf{Q} \mathbf{A} \mathbf{Q}^T \mathbf{n}$ over all possible $\mathbf{n}$ that is perpendicular to the fiber direction, with all $\mathbf{d}$ perpendicular to $\mathbf{n}$ and has unit length.
i.e.,
Maximize $\tilde{\mathbf{d}}^T \mathbf{A} \tilde{\mathbf{n}} + c_F \tilde{\mathbf{n}}^T \mathbf{A} \tilde{\mathbf{n}}$ over all possible $\mathbf{n}$ that is perpendicular to the fiber direction, with all $\mathbf{d}$ perpendicular to $\mathbf{n}$ and has unit length, where $\tilde{\mathbf{n}} = \mathbf{Q}^T \mathbf{n}$ and $\tilde{\mathbf{d}} = \mathbf{Q}^T \mathbf{d}$. Since in our discretization, the fiber direction is $\mathbf{q}_1$, therefore $\mathbf{q}_1^T \mathbf{n} = 0$, then we know $\tilde{\mathbf{n}} = \pm(0, 1)$
i.e.,
Maximize $\tilde{\mathbf{d}}^T \mathbf{A} \tilde{\mathbf{n}} + c_F \tilde{\mathbf{n}}^T \mathbf{A} \tilde{\mathbf{n}}$ over $\tilde{\mathbf{n}} = (0, \pm 1)$ and $\tilde{\mathbf{d}} = (\pm 1, 0)$
i.e.,

Maximize $\tilde{\mathbf{d}}^T\mathbf{A}\tilde{\mathbf{n}} + c_F\tilde{\mathbf{n}}^T\mathbf{A}\tilde{\mathbf{n}}$ over $\tilde{\mathbf{n}} = (0, \pm 1)$ and $\tilde{\mathbf{d}} = (\pm 1, 0)$
The maximum is

$$\pm g' r_{12} r_{22} + c_F h' r_{22}$$

Assume $g(x) = \gamma x$, $h(x) = \frac{1}{3}s(1-x)^3$ for $x \leq 1$, 0 otherwise.
When $r_{22} > 1$, $h = 0$, the maximum is

$$|\gamma r_{12} r_{22}|,$$

the return mapping is setting $r_{12} = 0$.
When $r_{22} < 1$, the maximum is

$$\pm\gamma |r_{12}| r_{22} - c_F s(1 - r_{22})^2 r_{22}$$

The yield surface is therefore

$$max(\pm\frac{\gamma}{s}|r_{12}|r_{22} - c_F(1 - r_{22})^2 r_{22}) \leq 0$$

If $r_{22} < 0$, the max should choose $-\frac{\gamma}{s}$. The return ampping is setting $r_{12} = 0$.
Otherwise, $r_{22} \in [0, 1]$, try to scale r_{12} to satisfy

$$\frac{\gamma}{s}|r_{12}| - c_F(1 - r_{22})^2 \leq 0$$

6 Derivative of $\hat{\mathbf{F}}^E$

In the paper we mention that $\frac{\partial \hat{\mathbf{F}}_p^E}{\partial \mathbf{x}_i}$ is a third order tensor, and does not depend on $\hat{\mathbf{x}}$ because $\hat{\mathbf{F}}_p^E$ is linear in $\hat{\mathbf{x}}_i$. Here we give the derivation of computing this derivative. We have by definition

$$\hat{\mathbf{d}}_{p,\beta}^E(\hat{\mathbf{x}}) = \hat{\mathbf{x}}_{\text{mesh}(p,\beta)} - \hat{\mathbf{x}}_{\text{mesh}(p,0)} \tag{2}$$

$$\hat{\mathbf{d}}_{p,\beta}^E(\hat{\mathbf{x}}) = (\nabla\hat{\mathbf{x}})_p \mathbf{d}_{p,\beta}^{E,n} \tag{3}$$

$$\hat{\mathbf{x}}_q = \sum_i \hat{\mathbf{x}}_i w_{iq}^n \tag{4}$$

$$\hat{\mathbf{F}}_p^E(\hat{\mathbf{x}}) = \hat{\mathbf{d}}_p^E \mathbf{D}_p^{-1} \tag{5}$$

Plugging in

$$\hat{F}_{p,\alpha\epsilon}^E = \sum_i \left(\sum_{\beta=1}^{\gamma} (w_{i,\text{mesh}(p,\beta)}^n - w_{i,\text{mesh}(p,0)}^n) \mathbf{x}_{i,\alpha} D_{p,\beta\epsilon}^{-1} + \sum_{\beta=\gamma+1}^{3} \sum_{\kappa=1}^{3} \frac{\partial w_{ip}^n}{\partial x_\kappa} d_{p,\kappa\beta}^{E,n} \mathbf{x}_{i,\alpha} D_{p,\beta\epsilon}^{-1} \right)$$

Differentiating we have

$$\frac{\partial \hat{F}_{p,\alpha\epsilon}^E}{\partial x_{i\zeta}} = \sum_{\beta=1}^{\gamma} (w_{i,\text{mesh}(p,\beta)}^n - w_{i,\text{mesh}(p,0)}^n) \delta_{\alpha\zeta} D_{p,\beta\epsilon}^{-1} + \sum_{\beta=\gamma+1}^{3} \sum_{\kappa=1}^{3} \frac{\partial w_{ip}^n}{\partial x_\kappa} d_{p,\kappa\beta}^{E,n} \delta_{\alpha\zeta} D_{p,\beta\epsilon}^{-1}$$

which doesn't depend on $\hat{\mathbf{x}}$. Note we can think of the undeformed segment or triangle as being aligned with the x-axis or xy-plane respectively. This will mean that the initial $\mathbf{F}$ is no longer $\mathbf{I}$, but rather the rotation which maps the axis aligned element to its initial position in space. This is not an issue as our elastic energy density ψ is world space rotation invariant. However this allows us to assume that $\mathbf{D}_p$ is block diagonal of the form

$$\mathbf{D}_p = \begin{bmatrix} D_{11} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for segments and

$$\mathbf{D}_p = \begin{bmatrix} D_{11} & D_{12} & 0 \\ 0 & D_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

for triangles. This saves memory for $\mathbf{D}_p$, and simplifies computing $\mathbf{D}_p^{-1}$.

7 Computation of Force on the Grid

$$\mathbf{f}_i^{(iii)}(\hat{\mathbf{x}}) = -\sum_{p\in\mathcal{I}^{(iii)}} V_p^0 \frac{\partial\psi}{\partial\mathbf{F}} : \frac{\partial\hat{\mathbf{F}}_p^E}{\partial\mathbf{x}_i} \tag{6}$$

$$\mathbf{f}_{i\zeta}^{(iii)}(\hat{\mathbf{x}}) = -\sum_{p\in\mathcal{I}^{(iii)}} \sum_\alpha \sum_\epsilon V_p^0 \frac{\partial\psi}{\partial F_{\alpha\epsilon}} \frac{\partial\hat{F}_{p,\alpha\epsilon}^E}{\partial x_{i\zeta}} \tag{7}$$

$$\mathbf{f}_{i\zeta}^{(iii)}(\hat{\mathbf{x}}) = -\sum_{p\in\mathcal{I}^{(iii)}} \sum_\epsilon V_p^0 \frac{\partial\psi}{\partial F_{\zeta\epsilon}} \Bigg(\sum_{\beta=1}^{\gamma} (w_{i,\text{mesh}(p,\beta)}^n - w_{i,\text{mesh}(p,0)}^n) D_{p,\beta\epsilon}^{-1} + \tag{8}$$

$$\sum_{\beta=\gamma+1}^{3} \sum_{\kappa=1}^{3} \frac{\partial w_{ip}^n}{\partial x_\kappa} d_{p,\kappa\beta}^{E,n} D_{p,\beta\epsilon}^{-1} \Bigg) \tag{9}$$

$$\mathbf{f}_{i\zeta}^{(iii)}(\hat{\mathbf{x}}) = -\sum_{p\in\mathcal{I}^{(iii)}} \sum_\epsilon V_p^0 \frac{\partial\psi}{\partial F_{\zeta\epsilon}} \Bigg(\sum_{\beta=1}^{\gamma} (w_{i,\text{mesh}(p,\beta)}^n - w_{i,\text{mesh}(p,0)}^n) D_{p,\beta\epsilon}^{-1} + \tag{10}$$

$$\sum_{\beta=\gamma+1}^{3} \sum_{\kappa=1}^{3} \frac{\partial w_{ip}^n}{\partial x_\kappa} d_{p,\kappa\beta}^{E,n} \delta_{\beta\epsilon} \Bigg) \tag{11}$$

Define

$$\mathbf{f}_{q\zeta}^{(ii)}(\hat{\mathbf{x}}) = -\sum_{p\in\mathcal{I}^{(iii)}} \sum_\epsilon V_p^0 \frac{\partial\psi}{\partial F_{\zeta\epsilon}} \sum_{\beta=1}^{\gamma} (\delta_{q,\text{mesh}(p,\beta)} - \delta_{q,\text{mesh}(p,0)}) D_{p,\beta\epsilon}^{-1} \tag{12}$$

$$= -\sum_{p\in\mathcal{I}^{(iii)}} \sum_{\epsilon=1}^{\gamma} V_p^0 \frac{\partial\psi}{\partial F_{\zeta\epsilon}} \sum_{\beta=1}^{\gamma} (\delta_{q,\text{mesh}(p,\beta)} - \delta_{q,\text{mesh}(p,0)}) D_{p,\beta\epsilon}^{-1} \tag{13}$$

Then

$$\mathbf{f}_{i\zeta}^{(iii)}(\hat{\mathbf{x}}) = \sum_{p\in\mathcal{I}^{(ii)}} \mathbf{f}_{q\zeta}^{(ii)}(\hat{\mathbf{x}}) w_{ip}^n - \sum_{p\in\mathcal{I}^{(iii)}} \sum_\epsilon V_p^0 \frac{\partial\psi}{\partial F_{\zeta\epsilon}} \sum_{\beta=\gamma+1}^{3} \sum_{\kappa=1}^{3} \frac{\partial w_{ip}^n}{\partial x_\kappa} d_{p,\kappa\beta}^{E,n} \delta_{\beta\epsilon} \tag{14}$$

$$\mathbf{f}_{i\zeta}^{(iii)}(\hat{\mathbf{x}}) = \sum_{p\in\mathcal{I}^{(ii)}} \mathbf{f}_{q\zeta}^{(ii)}(\hat{\mathbf{x}}) w_{ip}^n - \sum_{p\in\mathcal{I}^{(iii)}} \sum_{\epsilon=\gamma+1}^{3} V_p^0 \frac{\partial\psi}{\partial F_{\zeta\epsilon}} \sum_{\kappa=1}^{3} \frac{\partial w_{ip}^n}{\partial x_\kappa} d_{p,\kappa\epsilon}^{E,n} \tag{15}$$

8 Pseudocode

Algorithm 1 Simulate

procedure TIME_STEP
  TRANSFER_TO_GRID
  GRID_STEP
  TRANSFER_TO_PARTICLES
  UPDATE_PARTICLE_STATE
  PLASTICITY

procedure TRANSFER_TO_GRID
  for all grid nodes i **do**
    $m_i^n \leftarrow \sum_p w_{ip}^n m_p$
    $\mathbf{v}_i^n \leftarrow \frac{1}{m_i^n} \sum_p w_{ip}^n m_p \big(\mathbf{v}_p^n + \mathbf{C}_p^n(\mathbf{x}_i - \mathbf{x}_p^n)\big)$

procedure GRID_STEP
  $\langle \mathbf{v}_i^\star \rangle \leftarrow \langle \mathbf{v}_i^n \rangle +$ FORCE_INCREMENT$(\langle \mathbf{F}_p^{E,n} \rangle)$
  $\langle \overline{\mathbf{v}}_i^{n+1} \rangle \leftarrow$ GRID_COLLISIONS$(\langle \mathbf{v}_i^\star \rangle)$
  $\langle \tilde{\mathbf{v}}_i^{n+1} \rangle \leftarrow$ FRICTION$(\langle \overline{\mathbf{v}}_i^{n+1} \rangle, \langle \overline{\mathbf{v}}_i^{n+1} - \mathbf{v}_i^\star \rangle)$

procedure TRANSFER_TO_PARTICLES
  for all particles p of type (i) and (ii) **do**
    $\mathbf{v}_p^{n+1} \leftarrow \sum_i w_{ip}^n \tilde{\mathbf{v}}_i^{n+1}$
  for all particles p of type (iii) **do**
    $\mathbf{v}_p^{n+1} \leftarrow \sum_{\beta=0}^{\gamma} \frac{1}{\gamma} \mathbf{v}_{\mathrm{mesh}(p,\beta)}^{n+1}$
  for all particles p **do**
    $\mathbf{C}_p^{n+1} \leftarrow \sum_i w_{ip}^n \tilde{\mathbf{v}}_i^{n+1} (\mathbf{x}_i - \mathbf{x}_p^n)^T$

procedure UPDATE_PARTICLE_STATE
  for all particles p of type (i) **do**
    $\mathbf{x}_p^{n+1} \leftarrow \sum_i w_{ip}^n (\mathbf{x}_i^n + \Delta t \overline{\mathbf{v}}_i^{n+1})$
    $\nabla \mathbf{v}_p \leftarrow \sum_i \overline{\mathbf{v}}_i^{n+1} (\nabla w_{ip}^n)^T$
    $\hat{\mathbf{F}}_p^{E,n+1} \leftarrow (\mathbf{I} + \Delta t \nabla \mathbf{v}_p) \mathbf{F}_p^{E,n}$
  for all particles p of type (ii) **do**
    $\mathbf{x}_p^{n+1} \leftarrow \sum_i w_{ip}^n (\mathbf{x}_i^n + \Delta t \overline{\mathbf{v}}_i^{n+1})$
  for all particles p of type (iii) **do**
    $\mathbf{x}_p^{n+1} \leftarrow \sum_{\beta=0}^{\gamma} \frac{1}{\gamma} \mathbf{x}_{\mathrm{mesh}(p,\beta)}^{n+1}$
    $\nabla \mathbf{v}_p \leftarrow \sum_i \overline{\mathbf{v}}_i^{n+1} (\nabla w_{ip}^n)^T$
    for $\beta = 1$ to γ **do**
      $\hat{\mathbf{d}}_{p,\beta}^{E,n+1} \leftarrow \mathbf{x}_{\mathrm{mesh}(p,\beta)}^{n+1} - \mathbf{x}_{\mathrm{mesh}(p,0)}^{n+1}$
    for $\beta = \gamma + 1$ to 3 **do**
      $\hat{\mathbf{d}}_{p,\beta}^{E,n+1} \leftarrow (\mathbf{I} + \Delta t \nabla \mathbf{v}_p) \mathbf{d}_p^{E,n}$
    $\hat{\mathbf{F}}_p^{E,n+1} \leftarrow \hat{\mathbf{d}}_p^{E,n+1} \mathbf{D}_p^{-1}$

procedure PLASTICITY
  for all particles p of type (i) and (iii) **do**
    $\mathbf{F}_p^{E,n+1} \leftarrow$ RETURN_MAPPING$(\hat{\mathbf{F}}_p^{E,n+1})$