Research Statement

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My mathematical research primarily concerns descriptive set theory and its applications to combinatorics and theoretical computer science.

1 Introduction

1.1 Distributed and LOCAL Algorithms

Distributed computing involves the coordination of computers in a large network to solve a shared problem. As a prototypical example, consider the network of Wi-Fi routers in the United States. If all routers operate on the same channel, then a user connected to one router may experience interference from other nearby routers. Thus, if two routers are close together, they should operate on different channels.

There is a simple greedy algorithm for assigning channels to routers so that nearby routers do not interfere: Begin by enumerating the routers. Then, in order, each router selects the first channel not already chosen by any nearby routers. However, this algorithm is inefficient. If there are 300 million Wi-Fi routers in the United States, and each router takes ten seconds to determine the channels on which all nearby routers are operating, the greedy algorithm will run for nearly 100 years.

The reason for this algorithm's inefficiency is its *nonlocality*: Whether or not a specific channel is available to a given router may depend on the channel assignments of routers which are arbitrarily far away. It is therefore desirable to study **LOCAL algorithms**, in which each router's channel assignment depends only on other routers within a small radius. The study of LOCAL algorithms was prompted largely by the seminal 1992 paper of Linial ([Lin92]) and has since been continued by many theoretical computer scientists ([PS95], [BE13], [CLP20], [GHKM21]).

In the LOCAL setting, it is often convenient to adopt the language of graph theory. A computer network may be viewed as a graph in which the vertices are the computers and two vertices are adjacent if the computers they represent are close together; the channel assignment problem described above thus becomes a problem of producing a proper graph coloring, which is a coloring of the vertices of the graph such that adjacent vertices have different colors.

A **LOCAL coloring algorithm** for a collection $\mathcal{G}$ of finite graphs is a LOCAL algorithm $\mathcal{A}$ with the following property: Given a graph $G \in \mathcal{G}$, if each vertex in G outputs the color it computes by executing $\mathcal{A}$, then the coloring of G that results is proper. Note that each vertex executes *the same* algorithm. Therefore, it is necessary to break symmetry; this is typically done by initially assigning an identifier to each vertex using either a deterministic or a randomized procedure.

1.2 Descriptive Graph Combinatorics

Descriptive set theory is the study of definable subsets of Polish (i.e., separable and completely metrizable) topological spaces. The meaning of "definable" depends on the context; notions of definability which interest descriptive set theorists include Borel measurability, μ-measurability with respect to a (Borel) measure μ, and Baire measurability. A compelling motivation for studying descriptive set theory is that definable sets enjoy regularity properties not typically shared by non-definable sets.

A prominent subfield of descriptive set theory is **descriptive graph combinatorics**, the study of the definable combinatorics of definable graphs. Here, "definable graph" is typically synonymous with "**Borel graph**", where a graph is Borel if its vertex set is a Polish space X and its edge relation is Borel as a subset of X^2 with the product topology. Research on descriptive graph combinatorics began with the seminal 1999 paper of Kechris, Solecki, and Todorčević ([KST99]) and continues to be active today ([CM16], [Mar16], [CMT16], [CJM$^+$20], [CJM$^+$22], [GJKS23]).

Much of this body of work is dedicated to improving our understanding of the similarities and differences between classical (i.e., finitary) and definable combinatorics. In the finite setting, the notion of definability is not meaningful: Any finite set with the discrete topology is Polish, and any finite subset of a Polish space is Borel. Therefore, all finite graphs are Borel, and on finite graphs all standard combinatorial constructions – including vertex colorings, edge colorings, and perfect matchings – are definable. As a result, descriptive graph combinatorics is primarily concerned with infinite graphs.

1.3 Connections between Descriptive Combinatorics and LOCAL Algorithms

At first blush, descriptive set theory, which deals with infinite graphs, appears unrelated to distributed computing, which deals with finite graphs. Indeed, for two decades, the study of definable colorings remained largely separate from the study of LOCAL coloring algorithms. Then, in 2020, Bernshteyn revealed a profound connection between these two perspectives: Let $\mathcal{G}$ be a collection of finite graphs satisfying a mild closure condition, and let G be a bounded-degree Borel graph whose finite induced subgraphs all belong to $\mathcal{G}$. If there is a sufficiently local[1] deterministic (resp., randomized) algorithm for properly coloring the graphs in $\mathcal{G}$ with $d \in \mathbb{N}$ colors, then there is a Borel (resp., μ-measurable or Baire measurable) proper d-coloring of G ([Ber23]).

This result is an instance of a common mathematical theme; infinite limits of increasingly large finitary objects reflect the properties of their finite constituents. Work at the intersection of definable combinatorics and LOCAL algorithms has since resulted in many fruitful collaborations between descriptive set theorists and theoretical computer scientists ([BCG$^+$21], [BD23], [GR23]).

2 Research Direction #1: Descriptive Digraph Combinatorics

So far, most of the work within the intersection of descriptive combinatorics and LOCAL computing has focused on undirected graphs. However, relatively little is known con-

[1]The algorithm must be such that, when a graph G of size $n \in \mathbb{N}$ is given as input, each vertex needs to query only the vertices in its $o(\log(n))$-neighborhood to determine its color.

cerning **directed graphs** (or digraphs) in this context. A directed graph consists of a set of vertices and a set of directed edges, which are *ordered* pairs of vertices. Note that the edge relation of a digraph need not be symmetric. A vertex coloring c of a digraph is a **dicoloring** if there are no c-monochromatic directed cycles. This is a useful combinatorial notion that has been well-studied in the classical literature.

Currently, I am analyzing the combinatorics of digraphs from the perspectives of both descriptive set theory and LOCAL computing. This work is uncovering further connections between these two areas by exploring how they relate in a new context.

A natural starting point is to extend the classical combinatorial theorems on digraphs to the definable and LOCAL settings. Such extensions have been successful for undirected graphs. For instance, in 1941, Brooks classified the finite undirected graphs of degree bounded by $d \in \mathbb{N}$ which are not properly d-colorable: If $d = 2$, then the only such graphs are those containing odd cycles; if $d \geq 3$, then the only such graphs are those containing the complete graph on d vertices ([Bro41]). Today, due to work of Ghaffari, Hirvonen, Kuhn, and Maus ([GHKM21]), there is a highly local randomized algorithm for Brooks's theorem. Furthermore, there are μ-measurable and Baire measurable versions of Brooks's theorem due to work of Conley, Marks, and Tucker-Drob ([CMT16]).

In [HM11], Harutyunyan and Mohar prove a version of Brooks's theorem for digraphs. Here some additional terminology is needed. For any vertex v in a digraph D, the out-degree (resp., in-degree) of v is the number of out- (resp., in-) edges in D incident on v, and the max-degree of v is the maximum of its out-degree and its in-degree. A digraph is symmetric if, whenever there is an edge oriented from a vertex v to a vertex w, there is also an edge oriented from w to v.

Theorem 2.1 ([HM11], Theorem 2.1)**.** *Let $d \geq 1$. Suppose D is a digraph of max-degree bounded by d that is not d-dicolorable. If $d = 1$, then D contains a directed cycle; if $d = 2$, then D contains an odd symmetric cycle; and if $d \geq 3$, then D contains the complete symmetric digraph on d vertices.*

The proof of this result uses purely classical combinatorial techniques and cannot be directly adapted to the definable or LOCAL settings. Recently, however, I showed the following.

Theorem 2.2. *Let $d \geq 3$. There is a deterministic algorithm $\mathcal{A}$ that does the following: For each $n \in \mathbb{N}$, if D is a size-n digraph of max-degree bounded by d that does not contain the complete symmetric digraph on d vertices, then $\mathcal{A}$ returns a d-dicoloring of D such that each vertex needs to query only the vertices in its $O(d \cdot \mathsf{poly}(\log(n)))$-neighborhood.*

The proof combines the results of Harutyunyan and Mohar with a distributed layering technique which Ghaffari, Hirvonen, Kuhn, and Maus used to produce a deterministic LOCAL algorithm for the undirected version of Brooks's theorem ([GHKM21]). (The aforementioned randomized algorithm that appears in the same paper is, in a technical sense, "more local" than this deterministic algorithm.)

I believe the following (related) problems are now quite accessible.

Problem 2.3. *Let $d \geq 3$. Show that any Borel digraph on a Polish space X which does not contain the complete symmetric digraph on d vertices has, for each Borel probability measure μ on X, a proper μ-measurable d-dicoloring and also a proper Baire measurable d-dicoloring.*

Problem 2.4. *Let $d \geq 3$. Describe a randomized algorithm that, when given a size-n input digraph D that does not contain the complete symmetric digraph on d vertices, produces a d-dicoloring of D such that each vertex needs to query only the vertices in its $o(\log(n))$-neighborhood.*

There are two feasible approaches, and preliminary results indicate that both are fruitful. The first is to adapt the one-ended spanning forest technique used by Conley, Marks, and Tucker-Drob in [CMT16] to the digraph setting. The second is to apply the slack placement technique used by Ghaffari, Hirvonen, Kuhn, and Maus in [GHKM21] to digraphs.

Several other combinatorial problems – in particular, problems concerning edge colorings and perfect matchings – that interest theoretical computer scientists and descriptive set theorists alike appear to be both meaningful and tractable for digraphs. I plan to investigate these problems further in the near future.

3 Research Direction #2: The Complexity of Coloring Problems

In descriptive set theory, it is often desirable to determine the complexity of a combinatorial problem. In this context, the term "complexity" has a topological meaning. For instance, if the collection of graphs on which the combinatorial problem is solvable is an open set in some Polish space, then the problem is simple; if the collection of graphs on which the problem is solvable is an analytic non-Borel set, then the problem is complex. Complexity computations are motivated by the need to compare different problems with one another.

In their celebrated 2021 paper, Todorčević and Vidnyánszky prove the following complexity result.

Theorem 3.1 ([TV21], Theorem 1.3)**.** *The collection of acyclic Borel graphs which have proper Borel colorings with finitely many colors is $\mathbf{\Sigma}^1_2$-complete.*

A collection of sets that is $\mathbf{\Sigma}^1_2$-complete may be viewed as "difficult to describe". Therefore, this theorem of Todorčević and Vidnyánszky implies that it is hard to determine whether a given acyclic Borel graph has a proper Borel coloring with finitely many colors. The result has since been extended to show that even the smaller collection of *bounded-degree* acyclic Borel graphs which have proper Borel colorings with finitely many colors is $\mathbf{\Sigma}^1_2$-complete ([BCG+21]).

It is natural to inquire about the implications of these results for **hyperfinite graphs**. A Borel graph is hyperfinite if it is a countable increasing union of Borel graphs with finite connected components. Hyperfiniteness has long been an object of interest among descriptive set theorists; it is more complicated than finiteness but still tractable enough to permit structure theorems. The following is a deep open problem.

Problem 3.2. *Determine the complexity of the collection of bounded-degree acyclic hyperfinite Borel graphs which have proper Borel colorings with finitely many colors.*

While this problem seems difficult to attack directly, there are several important properties related to hyperfiniteness, and analyzing the combinatorial complexity of these properties may allow for inferences about the combinatorial complexity of hyperfiniteness. In a recent paper, Conley, Jackson, Marks, Seward, and Tucker-Drob introduce the following definition, which is a definable generalization of the classical notion of asymptotic dimension.

Definition 3.3 ([CJM+22], Definition 3.2). *Let G be a locally finite Borel graph on a standard Borel space X, and let ρ be the graph metric on G. Assume there exists $d \in \mathbb{N}$ such that, for all $r > 0$, there is a ρ-bounded Borel equivalence relation E on X such that, for all $x \in X$, the radius-r ball around x meets at most $d+1$ E-classes. Then the least such d is the* **Borel asymptotic separation index** *of G, denoted $asi_B(G)$.*

While currently no direct relationship between being hyperfinite and having Borel asymptotic separation index at most 1 is known, in the μ-measurable setting, these two properties are equivalent modulo a measure-zero set ([Wei21]). Therefore, the following problem may shed light on Problem 3.2.

Problem 3.4. *Determine the complexity of the collection of Borel graphs having Borel asymptotic separation index at most* 1*.*

Currently, Jan Grebík and I are pursuing the conjecture that the complexity is $\mathbf{\Sigma}^1_2$. This work is still at an early stage, but we have some preliminary results already that demonstrate a relationship between graphs that have Borel asymptotic separation index at most 1 and graphs on so-called "non-dominating sets", which are known to have high combinatorial complexity. We plan to continue this work and explore extensions to, for instance, bounded-degree graphs.

4 Future Directions

In addition to the future work I outlined above, I would also like to explore how Borel asymptotic separation index relates to other similar properties, including a strengthening known as Borel asymptotic dimension and a strong form of hyperfiniteness known as toast. I would also like to improve our current understanding of the relationship between combinatorial problems having Borel solutions and combinatorial problems that are solvable with LOCAL algorithms.

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