The Weird and Wonderful World of Large Cardinals

Cecelia Higgins

November 5, 2020
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To describe its construction, we need the following definitions.

- An ordinal is a transitive set that is well ordered by the membership relation $\in$.
- The collection of all ordinals is a proper class, denoted Ord.
- If $\alpha, \beta \in$ Ord, then we write $\alpha < \beta$ for $\beta \in \alpha$.
- For any ordinal $\alpha$, its ordinal successor $\alpha + 1$ is defined as $\{\alpha\}$.
- An ordinal is a limit ordinal if it is nonempty and is not a successor ordinal.

Examples

The first several ordinals are: $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$, $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$, ...

To set theorists, these ordinals are, respectively, the natural numbers 0, 1, 2, ...

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The Universe (No, Not the Physical One)

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An **ordinal** is a transitive set that is well ordered by the membership relation \( \in \). The collection of all ordinals is a proper class, denoted \( \text{Ord} \). If \( \alpha, \beta \in \text{Ord} \), then we write \( \alpha < \beta \) for \( \alpha \in \beta \).
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**Definition**

The universe is constructed in stages as follows:

\[ V_0 = \emptyset, \]

For all ordinals \( \alpha \), \( V_{\alpha+1} = \mathcal{P}(V_\alpha) \),

For all limit ordinals \( \lambda \), \( V_\lambda = \bigcup_{\alpha < \lambda} V_\alpha \),

\[ V = \bigcup_{\alpha \in \text{Ord}} V_\alpha. \]
Let’s focus our attention on one particular stage of $V$ – the $\omega$th stage, $V_\omega$. 
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**Fact**

The set of all hereditarily finite sets is precisely $V_\omega$. 

If you are an ordinal living in $V_\omega$, the world of hereditarily finite sets, what does $\omega$ “look like” to you? Perhaps a better question is: How might you (and your other finite ordinal friends) try and “reach” $\omega$?
Life in the World of Hereditarily Finite Sets

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If you are an ordinal living in $V_\omega$, the world of hereditarily finite sets, what does $\omega$ “look like” to you? Perhaps a better question is: How might you (and your other finite ordinal friends) try and “reach” $\omega$?

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Thus, we may view $\omega$ as being “inaccessible” from $V_\omega$. 
We now isolate the special properties of $\omega$ discussed on the previous slide.
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**Definitions**

An ordinal $\kappa$ is a **cardinal** if, for each ordinal $\alpha < \kappa$, there does not exist a surjection $f : \alpha \rightarrow \kappa$. 

These properties correspond to axioms of ZFC; strong limitness corresponds to the power set axiom, and regularity corresponds to the replacement scheme. These are two of the “harder” axioms to ask a stage $V^{\kappa}$ of $V$ to satisfy; under weaker assumptions on $\kappa$ (e.g. uncountability), all the other axioms of ZFC hold in $V^{\kappa}$. Therefore, if $\kappa$ is an uncountable regular strong limit cardinal, then $V^{\kappa}$ is a model of ZFC.
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Therefore, if $\kappa$ is an uncountable regular strong limit cardinal, then $V_\kappa$ is a model of ZFC.
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The Smallest Large Cardinals

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**Definition**

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So strongly inaccessible cardinals are large cardinals. There are many more large cardinal properties, many of which are of great interest to set theorists.
Set theorists care about large cardinals because:
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- They are the focal point of the inner model problem, one of the biggest open problems in the field today.

- They have important implications for determinacy.

- They have meaningful interactions with forcing.

Here are some areas of math outside set theory for which large cardinals have implications:

- **Algebraic topology**: Vopenka’s principle, which is considered a large cardinal principle. (See “Implications of large-cardinal principles in homotopical localization”, by Casacuberta, Scevenels, and Smith.)

- **Measure theory**: Measurable cardinals.
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**Some Motivation**

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- Algebraic topology: Vopěnka’s principle, which is considered a large cardinal principle. (See “Implications of large-cardinal principles in homotopical localization”, by Casacuberta, Scevenels, and Smith.)
- Measure theory: Measurable cardinals.
For the remainder of this talk, we'll focus on proving a beautiful theorem about large cardinals known as Scott’s theorem.
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**Theorem (Scott)**

If there exists a measurable cardinal, then $V \neq L$. 
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**Definitions**

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Where We’re Headed

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**Definition**

A cardinal $\kappa$ is **measurable** if it is uncountable and there exists a nonprincipal, $\kappa$-complete ultrafilter on $\kappa$.

The crux of the proof of Scott’s theorem will be to show that any measurable cardinal is the critical point of an elementary embedding of the universe $V$ into some inner model.
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**Definition**

Let $I$ be a set. A set $\mathcal{U} \subseteq \mathcal{P}(I)$ is a filter on $I$ if:

1. $\emptyset \notin \mathcal{U}$ and $I \in \mathcal{U}$,
2. If $X \in \mathcal{U}$ and $X \subseteq Y \subseteq I$, then $Y \in \mathcal{U}$,
3. If $X, Y \in \mathcal{U}$, then $X \setminus Y \in \mathcal{U}$.

The ultrafilter $\mathcal{U}$ is an ultrafilter if, for all $X \subseteq I$, either $X \in \mathcal{U}$ or $I \setminus X \in \mathcal{U}$.

The ultrafilter $\mathcal{U}$ is $\mathfrak{c}$-complete for a cardinal $\mathfrak{c}$ if, for all cardinals $\mathfrak{a} < \mathfrak{c}$, if $\{A_\alpha : \alpha < \mathfrak{a}\}$ is a collection of sets in $\mathcal{U}$, then $\bigcup \{A_\alpha : \alpha < \mathfrak{a}\} \in \mathcal{U}$. 
Measurables and Ultrafilters and Embeddings (Oh My!)

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The filter \( U \) is an **ultrafilter** if, for all \( X \subseteq I \), either \( X \in U \) or \( I \setminus X \in U \).

The ultrafilter \( U \) is **principal** if there exists some \( X \subseteq I \) such that \( U = \{ Y \subseteq I : X \subseteq Y \} \).
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The ultrafilter $U$ is $\kappa$-complete for a cardinal $\kappa$ if, for all cardinals $\lambda < \kappa$, if $\{ A_\alpha : \alpha < \lambda \}$ is a collection of sets in $U$, then $\bigcap_{\alpha<\lambda} A_\alpha \in U$. 

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An elementary embedding is a truth-preserving injection between inner models.

Definition

Let $M_1$, $M_2$ be inner models. Then the class function $j: M_1 \rightarrow M_2$ is an elementary embedding if, for all formulas $\varphi(x_1, \ldots, x_n)$ in the language of set theory and for all sets $a_1, \ldots, a_n \in M_1$, $M_1 \models \varphi(a_1, \ldots, a_n)$ if and only if $M_2 \models \varphi(j(a_1), \ldots, j(a_n))$. 

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**Definition**

If $j : V \rightarrow M$ is a nontrivial elementary embedding, then the least ordinal moved by $j$ is called the **critical point** of $j$. 
An Ultra-Powerful Construction

To prove that measurable cardinals are the critical points of elementary embeddings, we need to find an appropriate inner model in which to embed $V$. To that end, we will consider the following important construction.
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Consider the proper class $C$ consisting of all functions having domain
$$\kappa = \{\alpha \in \text{Ord} : \alpha < \kappa\}.$$
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$$[f]_\mathcal{U} = \{g \in C : g =^* f \text{ and } \forall h (h =^* f \rightarrow \text{rank}(h) \geq \text{rank}(g))\}.$$
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$$[f]_\mathcal{U} = \{ g \in C : g \equiv^* f \text{ and } \forall h (h \equiv^* f \rightarrow \text{rank}(h) \geq \text{rank}(g)) \}.$$

Define also a binary relation $\in^*$ on the collection of modified equivalence classes by $[f]_\mathcal{U} \in^* [g]_\mathcal{U}$ if $\{ \alpha < \kappa : f(\alpha) \in g(\alpha) \} \in \mathcal{U}$.

The ultrapower of $V$ by $\mathcal{U}$, denoted $\text{Ult}(V, \mathcal{U})$, is the proper class consisting of the modified equivalence classes $[f]_\mathcal{U}$ for all $f \in C$, equipped with the binary relation $\in^*$. 
We would like to view $\text{Ult}(V, U)$ as an inner model. However, inner models must be **transitive**.
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**There is a transitive proper class $M$ isomorphic to $\text{Ult}(\mathcal{V}, \mathcal{U})$.**
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Using $\kappa$-completeness of $U$, we can show that $\in^*$ is a well-founded relation. There is a theorem of set theory, called the Mostowski collapsing lemma, that therefore enables us to conclude the following:

**There is a transitive proper class $M$ isomorphic to Ult($V, U$).**

The isomorphism $\pi : \text{Ult}(V, U) \rightarrow M$ can be defined by transfinite recursion as follows:

$$\pi([f]) = \{\pi([g]) : [g] \in^* [f]\}.$$
Now, we’ll construct the elementary embedding $j : V \rightarrow M$. 
Now, we’ll construct the elementary embedding $j : V \to M$.

For each set $a$, we define the constant function on $a$, denoted $c_a$, to be the function defined on $\kappa$ by $c_a(\alpha) = a$ for all $\alpha < \kappa$. 
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So let’s define $j : V \to M$ by $j(a) = \pi([c_a])$. 
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So let’s define $j : V \rightarrow M$ by $j(a) = \pi([c_a])$. We claim that $j$ is a nontrivial elementary embedding with critical point $\kappa$. 
To see that $j$ is an elementary embedding, we’ll need the following:

**Theorem (Łoś)**

Let $\varphi(x_1, \ldots, x_n)$ be any formula in the language of set theory, and let $f_1, \ldots, f_n$ be functions with domain $\mathbb{V}$. Then $\text{Ult}(V, U)|= \varphi(f_1, \ldots, f_n)$ if $\{\varphi(<\mathbb{V} >): \varphi(f_1(<\mathbb{V} >), \ldots, f_n(<\mathbb{V} >))}\in U$.

Since $\text{Ult}(V, U)$ is isomorphic to $M$ via $\varphi$, we have as a consequence $M|= \varphi(\varphi(f_1), \ldots, \varphi(f_n))$ if $\{\varphi(<\mathbb{V} >): \varphi(c_{a_1}, \ldots, c_{a_n})\}\in U$.

So, let $a_1, \ldots, a_n$ be sets. Then $M|= \varphi(j(a_1), \ldots, j(a_n))$ if $M|= \varphi(\varphi(c_{a_1}), \ldots, \varphi(c_{a_n}))$ if $\{\varphi(<\mathbb{V} >): \varphi(a_1, \ldots, a_n)\}\in U$.
To see that \( j \) is an elementary embedding, we’ll need the following:

**Theorem (Łoś)**

Let \( \varphi(x_1, \ldots, x_n) \) be any formula in the language of set theory, and let \( f_1, \ldots, f_n \) be functions with domain \( \kappa \). Then

\[
\text{Ult}(V, \mathcal{U}) \models \varphi([f_1], \ldots, [f_n]) \iff \{ \alpha < \kappa : \varphi(f_1(\alpha), \ldots, f_n(\alpha)) \} \in \mathcal{U}.
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Since \( \text{Ult}(V, \mathcal{U}) \) is isomorphic to \( M \) via \( \pi \), we have as a consequence

\[
M \models \varphi(\pi([f_1]), \ldots, \pi([f_n])) \iff \{ \alpha < \kappa : \varphi(f_1(\alpha), \ldots, f_n(\alpha)) \} \in \mathcal{U}.
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The Elementary Embedding

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Let \( \varphi(x_1, \ldots, x_n) \) be any formula in the language of set theory, and let \( f_1, \ldots, f_n \) be functions with domain \( \kappa \). Then

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- Since \( \text{Ult}(V, \mathcal{U}) \) is isomorphic to \( M \) via \( \pi \), we have as a consequence

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\]

- So, let \( a_1, \ldots, a_n \) be sets. Then

\[
M \models \varphi(j(a_1), \ldots, j(a_n)) \iff \begin{align*}
& M \models \varphi(\pi([c_{a_1}]), \ldots, \pi([c_{a_n}])) \\
& \text{iff} \quad \{ \alpha < \kappa : \varphi(c_{a_1}(\alpha), \ldots, c_{a_n}(\alpha)) \} \in \mathcal{U} \\
& \text{iff} \quad \{ \alpha < \kappa : \varphi(a_1, \ldots, a_n) \} \in \mathcal{U} \\
& \text{iff} \quad \varphi(a_1, \ldots, a_n).
\end{align*}
\]
Next, we’ll show that $j$ does not move any ordinals beneath $\kappa$. 
Next, we’ll show that \( j \) does not move any ordinals beneath \( \kappa \). Let’s fix an ordinal \( \alpha < \kappa \) and assume that, for all \( \beta < \alpha \), \( j(\beta) = \beta \).

We have
\[
j(\alpha) = \pi(\langle 0, \beta, \pi(\langle 0, \beta, \rangle) \rangle)
= \{ \pi(\langle 0, \beta, \rangle) : [\alpha] \in [\kappa]_j \}.
\]

If \( [\alpha] \in [\kappa]_j \), then
\[
\{ \sigma < n : \tau(\sigma) \in 2^\beta \} \subseteq m.
\]

So
\[
\bigcup \{ \sigma < n : \tau(\sigma) \in 2^\beta \} \subseteq m.
\]

By \( \kappa \)-completeness, there is some \( \delta < \delta \) such that
\[
\{ \sigma < n : \tau(\sigma) = 2^\beta \} \subseteq m.
\]

So \([\alpha] \in [\kappa]_j\).

Thus
\[
\pi(\langle 0, \beta, \rangle) = \pi(\langle 0, \beta, \rangle) = \delta \text{ since } \delta < \kappa.
\]
Finally, we’ll show that $j(\kappa) \neq \kappa$.

Define $d : \kappa \to \kappa$ by $d(\gamma) = \gamma$. Then for each $\gamma < \kappa$,

$$\{ \xi < \kappa : \gamma < d(\xi) \}$$

is unbounded.

Since $\kappa$ is $\kappa$-complete and nonprincipally

$$\{ \xi < \kappa : \gamma < d(\xi) \}$$

is ill-founded,

so $[\gamma] \in \mathcal{F}([d])$. Thus, for all $\gamma < \kappa$,

$$\pi([\gamma]) = \gamma \in \pi([d]).$$

So $\kappa \leq \pi([d])$. However, we also have

$$\{ \xi < \kappa : d(\xi) < \kappa \}$$

is ill-founded,

so $[d] \in \mathcal{F}([\kappa])$. Thus

$$\pi([d]) \subseteq \pi([\kappa]) = j(\kappa).$$

Then $\kappa < j(\kappa)$. 

We can now give the proof of Scott’s theorem in just a few lines.

To show: If there exists a measurable cardinal, then $V = L$.

Assume for contradiction that a measurable exists but $V = L$. Let $\kappa$ be the least measurable, and let $\mathcal{U}_\kappa = (V, \kappa)$, $M$, and $j: V \to M$ be as before.

Then since $V = \kappa$ is the least measurable, we have $M = j(\kappa)$ is the least measurable.

But by minimality of $L$, $V = M$. So $V = j(\kappa)$ is the least measurable.

But $j(\kappa) > \kappa$.