MEASURABLE BROOKS'S THEOREM FOR DIRECTED GRAPHS

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Abstract. We prove a descriptive version of Brooks's theorem for directed graphs. In particular, we show that, if D is a Borel directed graph on a standard Borel space X such that the maximum degree of each vertex is at most $d \geq 3$, then unless D contains the complete symmetric directed graph on $d+1$ vertices, D admits a μ-measurable d-dicoloring with respect to any Borel probability measure μ on X, and D admits a τ-Baire-measurable d-dicoloring with respect to any Polish topology τ compatible with the Borel structure on X. We also prove a definable version of Gallai's theorem on list dicolorings for directed graphs by showing that any Borel directed graph of bounded degree whose connected components are not Gallai trees is Borel degree-list-dicolorable.

1. Introduction

A classical graph theory result states that, for a finite undirected graph G such that each vertex has degree at most d, there is a proper $(d+1)$-coloring of G. The proof is a simple greedy algorithm argument; each vertex receives the first color that has not already been assigned to any of its neighbors. It is easy to see that this upper bound is sharp: For $d = 2$, if G is an odd cycle, then each vertex of G has degree 2, but the chromatic number of G is 3; for $d \geq 3$, if G is the complete graph on $d+1$ vertices, then each vertex of G has degree d, but the chromatic number of G is $d+1$.

However, these obvious obstructions to a smaller upper bound on the chromatic number are the only obstructions. A 1941 theorem of Brooks characterizes the graphs such that each vertex has degree at most d which have proper d-colorings as exactly the graphs which do not contain odd cycles or complete graphs.

Theorem 1.1 ([Bro41]). *Let G be a finite undirected graph such that each vertex has degree at most d. If $d = 2$, assume G has no odd cycles; if $d \geq 3$, assume G does not contain the complete graph on $d+1$ vertices. Then there is a proper d-coloring of G.*

Suppose we wish to impose an additional requirement on the proper coloring: Consider a function L which assigns to each vertex x of G a list $L(x)$ of colors. An *L-list-coloring* of G is a proper coloring c of G such that $c(x) \in L(x)$ for all vertices x of G. We say that G is *degree-list-colorable* if, for any function L such that $|L(x)| \geq \deg(x)$ for all vertices x, there is an L-list coloring of G. Note that, when $L(x) = \{0, 1, \ldots, d-1\}$ for all vertices x of G, then G is L-list-colorable if and only if G has a proper d-coloring.

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Recall that a set S of vertices of G is *biconnected* if the induced subgraph $G[S]$ of G is connected and the induced subgraphs $G[S \setminus \{s\}]$ are connected for each $s \in S$. A *block* in G is a maximal biconnected set of vertices of G. A *Gallai tree* is a connected graph in which each block induces either an odd cycle or a complete graph. Results obtained independently by Borodin and Erdős, Rubin, and Taylor characterize the finite undirected graphs which are degree-list-colorable as the graphs whose connected components are not Gallai trees. The following theorem is typically referred to as Gallai's theorem.

Theorem 1.2 ([Bor77], [ERT79])**.** *Let G be a finite undirected graph. Then G is degree-list-colorable if and only if no connected component of G is a Gallai tree.*

Brooks's proof of Theorem 1.1 and the proofs of Borodin and Erdős, Rubin, and Taylor of Theorem 1.2 all use classical techniques and do not immediately generalize to the descriptive setting. Indeed, results of Marks ([Mar16]) demonstrate that Brooks's theorem fails in the Borel context. However, in 2016, Conley, Marks, and Tucker-Drob gave both a measurable version of Brooks's theorem and a definable version of Gallai's theorem.

Theorem 1.3 ([CMT16], Theorem 1.2)**.** *Let G be an undirected Borel graph on a standard Borel space X such that each vertex has degree at most $d \geq 3$. Assume G does not contain the complete graph on $d+1$ vertices. Then:*

(1) For any Borel probability measure μ on X, there is a μ-measurable proper d-coloring of G.
(2) For any Polish topology τ compatible with the Borel structure on X, there is a τ-Baire-measurable proper d-coloring of G.

Theorem 1.4 ([CMT16], Theorem 1.4)**.** *Let G be a locally finite undirected Borel graph on a standard Borel space X. Assume that no connected component of G is a Gallai tree. Then G is Borel degree-list-colorable.*

In this paper, we show that the two theorems above have analogues for *directed graphs* (or *digraphs*), in which each edge between vertices has an orientation. The coloring problem of interest in the digraph setting is the problem of producing a *dicoloring*, an assignment of colors to vertices so that no *directed cycle*, that is, a sequence $(x_0, x_1, \ldots, x_k = x_0)$ of vertices in which x_i has an out-oriented edge to x_{i+1} for each $i < d$, is monochromatic. The notion of dicoloring was introduced first by Neumann-Lara ([NL82]) in 1982 and again by Mohar ([Moh03]) in 2003, and it has since been studied extensively.

Consider a digraph D such that each vertex has *maximum degree* at most d, so that each vertex has at most d out-oriented neighbors and at most d in-oriented neighbors. Then a simple greedy algorithm argument demonstrates that the *dichromatic number* of D, which is the least number of colors needed to produce a dicoloring of D, is less than or equal to $d+1$ (see [AA22] for a complete proof). As in the undirected setting, this bound is sharp: For $d = 1$, if D is a directed cycle, then the maximum degree of each vertex of D is 1, but the dichromatic number of D is 2; for $d = 2$, if D is the *symmetrization* of an undirected odd cycle – that is, if D is obtained by replacing each edge of an undirected odd cycle with both an out-oriented edge and an in-oriented edge – then each vertex of D has maximum degree 2, but the dichromatic number of D is 3; and for $d \geq 3$, if D is the symmetrization of the complete graph on $d+1$ vertices, then each vertex of D has maximum degree d, but the dichromatic number of D is $d+1$.

The following theorem, which was first stated by Mohar in 2010 and later proved in full by Harutyunyan and Mohar in 2011, parallels Brooks's theorem by characterizing the obstructions to having a smaller upper bound on the dichromatic number.

Theorem 1.5 ([Moh10], Theorem 2.3; [HM11])**.** *Let D be a finite directed graph such that each vertex has maximum degree at most d. If $d = 1$, assume D has no directed cycles; if $d = 2$, assume D does not contain the symmetrization of any undirected odd cycles; and if $d \geq 3$, assume D does not contain the symmetrization of the undirected complete graph on $d + 1$ vertices. Then there is a d-dicoloring of D.*

Our main result is a measurable version of this theorem. In particular, we show the following.

Theorem 1.6. *Let D be a Borel directed graph on a standard Borel space X. Suppose there is $d \geq 3$ such that the maximum degree of x is at most d for all $x \in X$, and assume D does not contain the symmetrization of the undirected complete graph on $d + 1$ vertices. Then:*

(1) For any Borel probability measure μ on X, there is a μ-measurable d-dicoloring of D.
(2) For any Polish topology τ compatible with the Borel structure on X, there is a τ-Baire-measurable d-dicoloring of D.

Also in 2011, Harutyunyan and Mohar proved a digraph version of Gallai's theorem. If L is a function which assigns to each vertex x of D a list $L(x)$ of colors, then an *L-list-dicoloring* of D is a dicoloring c of D such that $c(x) \in L(x)$ for all vertices x. We say that D is *degree-list-dicolorable* if, for any function L such that $|L(x)|$ is greater than or equal to the maximum degree of x for all vertices x of D, there is an L-list-dicoloring of D. In the digraph context, a *Gallai tree* is a connected digraph each of whose blocks induces a dicycle, the symmetrization of an undirected odd cycle, or the symmetrization of an undirected complete graph. A digraph is then degree-list-dicolorable if none of its connected components is a Gallai tree.

Theorem 1.7 ([HM11], Theorem 2.1)**.** *Let D be a finite directed graph. If no connected component of D is a Gallai tree, then D is degree-list-dicolorable.*

We prove the following definable version of this result for digraphs of bounded degree.

Theorem 1.8. *Let D be a Borel directed graph of bounded degree on a standard Borel space X. Assume that no connected component of D is a Gallai tree. Then D is Borel degree-list-dicolorable.*

In future work, we would like to better understand the implications of this result for descriptive digraph combinatorics more generally. In particular, although the problem of producing a dicoloring is not a locally checkable labeling problem in general, we are interested in studying possible connections between descriptive digraph combinatorics and LOCAL algorithms; this may lead to extensions of Bernshteyn's work ([Ber23]).

2. Preliminaries

We begin by recalling the basic terminology and notation for directed graphs, some of which was mentioned already in the introduction. A *directed graph* (or *digraph*) D is a pair $D = (X, A)$, where X is a set, called the *vertex set*, and $A \subseteq X^2$ is an irreflexive relation. The elements of A are called *arcs*.

Throughout the rest of this section, let $D = (X, A)$ be a directed graph on a set X. Let $x \in X$. An element $y \in X$ is an *out-neighbor* of x if there is an arc from x to y. The set of out-neighbors of x is denoted $N^+(x)$. An element $y \in X$ is an *in-neighbor* of x if there is an arc from y to x. The set of in-neighbors of x is denoted $N^-(x)$.

The *out-degree* of x, denoted $d^+(x)$, is defined by $d^+(x) = |N^+(x)|$. The *in-degree* of x, denoted $d^-(x)$, is defined by $d^-(x) = |N^-(x)|$. The *maximum degree* of x, denoted $d^{\max}(x)$, is defined by $d^{\max}(x) = \max\{d^+(x), d^-(x)\}$. The *minimum degree* of x, denoted $d^{\min}(x)$, is defined by $d^{\min}(x) = \min\{d^+(x), d^-(x)\}$. Note that it is possible that $N^+(x) \cap N^-(x) \neq \emptyset$.

The *maximum side* of x, denoted $N^{\max}(x)$, is defined by $N^{\max}(x) = N^+(x)$ if $|N^+(x)| \geq |N^-(x)|$ and $N^{\max}(x) = N^-(x)$ otherwise. The *minimum side* of x, denoted $N^{\min}(x)$, is defined by $N^{\min}(x) = N^+(x)$ if $|N^+(x)| \leq |N^-(x)|$ and $N^{\min}(x) = N^-(x)$ otherwise.

Let $D' = (X', A')$ be a *sub-digraph* of D, that is, a digraph such that $X' \subseteq X$ and $A' \subseteq A$. Then for each $x \in A'$, we write $d^+_{D'}(x)$ for the out-degree of x in D', $d^-_{D'}(x)$ for the in-degree of x in D', $d^{\max}_{D'}(x)$ for the maximum degree of x in D', and $d^{\min}_{D'}(x)$ for the minimum degree of x in D'.

A *directed cycle* (or *dicycle*) in D is a set $C = (x_0, x_1, \dots, x_k)$ of vertices of D such that $x_0 = x_k$, there is an arc from x_i to x_{i+1} for each $i < k$, and $x_i \neq x_j$ for all $i, j < k$ with $i \neq j$. If $k = 2$, then C is called a *digon*. A d*-dicoloring* of D is a function $c : X \to \{0, 1, \dots, d-1\}$ such that, for any dicycle C in D, there are points $x, x' \in C$ such that $c(x) \neq c(x')$. The *dichromatic number* of D, denoted $\vec{\chi}(D)$, is the least $d \in \mathbb{N}$ such that there is a d-dicoloring of D.

Let Y be a set, and let $[Y]^{<\infty}$ denote the collection of finite subsets of Y. A *(*Y*-)list assignment* is a function $L : X \to [Y]^{<\infty}$. Then D is L*-(list-)dicolorable* if there is a dicoloring c of D such that $c(x) \in L(x)$ for each $x \in X$. We say that D is *degree-list-dicolorable* if, for each list assignment L such that $|L(x)| \geq d^{\max}(x)$ for all $x \in X$, D is L-dicolorable, and we say that D is *(degree-plus-one)-list-dicolorable* if, for each list assignment L such that $|L(x)| > d^{\max}(x)$ for all $x \in X$, D is L-dicolorable.

Given an undirected graph G, we may replace each edge of G with a digon to obtain a directed graph. Precisely, the digraph $D = (X, A)$ is the *symmetrization* of the undirected graph $G = (X, E)$ if, for all $x, y \in X$, $\{x, y\} \in E$ if and only if $(x, y) \in A$ and $(y, x) \in A$. Note that, if D is the symmetrization of G, then $\vec{\chi}(D) = \chi(G)$. The digraph D is a *symmetric cycle* if D is the symmetrization of an undirected cyclic graph. The graph D is a *complete symmetric digraph* if it is the symmetrization of an undirected complete graph.

Conversely, by forgetting the orientations of the arcs in D, we obtain an undirected graph, called the *underlying graph* $\tilde{D}$ of D. The vertex set of $\tilde{D}$ is X, and two vertices $x, y \in X$ are adjacent in $\tilde{D}$ if and only if either $(x, y) \in A$ or $(y, x) \in A$.

Many definitions from the theory of undirected graphs may now be imported to the digraph context. In particular, we say that D is *locally finite* if $\tilde{D}$ is locally finite, D is *locally countable* if $\tilde{D}$ is locally countable, and D is *of bounded degree* if $\tilde{D}$ is of bounded degree. Also, we say D is *connected* if $\tilde{D}$ is connected. If $S \subseteq X$ and $|S| \geq 2$, then S is *biconnected* if the induced sub-digraph $\tilde{D}[S]$ is connected and the induced sub-digraphs $\tilde{D}[S \setminus \{s\}]$ are connected for all $s \in S$. A *block* in D is a maximal biconnected set in D. A connected component C of D is a *Gallai tree* if each block in $D[C]$ induces a dicycle, a (finite) odd symmetric cycle, or a (finite) complete symmetric digraph.

Throughout, we consider descriptive versions of the combinatorial notions for digraphs. For the fundamentals of descriptive set theory, we refer the reader to [Kec95].

Definition 2.1. Let X be a standard Borel space. A **Borel digraph** on X is a digraph $D = (X, A)$, where A is Borel in the product topology on X^2. If Y is a Polish space, then a dicoloring $c : X \to Y$ of D is a **Borel dicoloring** if c is a Borel function. The **Borel dichromatic number** of D, denoted $\vec{\chi}_B(D)$, is the least $d \in \mathbb{N}$ such that there is a Borel d-dicoloring of D.

A **Borel (Y-)list assignment** is a Borel function $L : X \to [Y]^{<\infty}$. We say that D is **Borel degree-list-dicolorable** if, for any Borel list assignment L such that $|L(x)| \geq d^{\max}(x)$ for all $x \in X$, D is Borel L-dicolorable. We say that D is **Borel (degree-plus-one)-list-dicolorable** if, for any Borel list assignment L such that $|L(x)| > d^{\max}(x)$ for all $x \in X$, D is Borel L-dicolorable.

3. Definable Digraph Combinatorics

In this section, we prove some initial results on the definable combinatorics of digraphs. We show first that, if the minimum degree of each vertex in a locally countable digraph is at most d, then the Borel dichromatic number is at most $d+1$. The proof of the following proposition is similar to the proof of Proposition 4.6 in [KST99].

Proposition 3.1. *Let $d \in \mathbb{N}$, and let D be a locally countable Borel digraph on a standard Borel space X. Suppose $d^{\min}(x) \leq d$ for all $x \in X$. Then $\vec{\chi}_B(D) \leq d+1$.*

Proof. We proceed by induction on d. Suppose $d = 0$; then D has no dicycles, and so clearly there is a Borel dicoloring of D with just one color.

Now let $d \geq 0$. Assume that, for any locally countable Borel digraph D' on a standard Borel space X', if $d^{\min}(x) \leq d$ for all $x \in X'$, then $\vec{\chi}_B(D') \leq d+1$. Suppose now $d^{\min}(x) \leq d+1$ for all $x \in X$. Since the underlying graph $\tilde{D}$ of D is locally countable, it follows from Proposition 4.3 in [KST99] that there is a countable Borel proper coloring c of $\tilde{D}$. Now define $A_0 = \{x \in X : c(x) = 0\}$, and, having defined A_n, let $A_{n+1} = \{x \in X : c(x) = n+1 \text{ and } N^{\min}(x) \cap (A_0 \cup \cdots \cup A_n) = \emptyset\}$. Write $A = \bigcup_{n \in \mathbb{N}} A_n$. Then A is Borel. We claim A contains no dicycles. Otherwise, let $C = (x_0, x_1, \ldots, x_k = x_0)$ be a dicycle in A. If $k = 2$, then assume without loss of generality that $c(x_0) > c(x_1)$. Then $N^{\min}(x_0) \cap A_{c(x_1)} \neq \emptyset$, so that $x_0 \notin A_{c(x_0)}$, a contradiction. If $k > 2$, then let $\alpha = \max\{c(x_0), c(x_1), \ldots, c(x_{k-1})\}$. Then for some $i < k$, $c(x_i) = \alpha$. If $N^{\min}(x_i) = N^+(x_i)$, then since $c(x_i) > c(x_{i+1})$ and since $x_{i+1} \in A_{c(x_{i+1})}$, $x_i \notin A_{c(x_i)}$, a contradiction. Similarly, if $N^{\min}(x_i) = N^-(x_i)$, then since $c(x_i) > c(x_{i-1})$ and since $x_{i-1} \in A_{c(x_{i-1})}$, $x_i \notin A_{c(x_i)}$.

We claim also that, for each $x \in X$, either $x \in A$ or there is $y \in N^{\min}(x) \cap A$. If $x \notin A$, then $x \notin A_{c(x)}$, so $c(x) > 0$ and $N^{\min}(x) \cap (A_0 \cup \cdots \cup A_{c(x)-1}) \neq \emptyset$. Therefore, $d^{\min}_{D[X \setminus A]}(x) \leq d$. So by the inductive hypothesis, there is a Borel dicoloring $c' : (X \setminus A) \to \{0, \ldots, d\}$ of $D[X \setminus A]$. Now define a Borel dicoloring c_0 of D by $c_0(x) = c'(x)$ if $x \in X \setminus A$ and $c_0(x) = d+1$ if $x \in A$. $\square$

Next, we show that any locally finite Borel digraph is Borel (degree-plus-one)-list-dicolorable. For the proof, we first separate the vertex set of the underlying graph into countably many independent sets $A_0, A_1, \dots$. Then we list-dicolor the independent sets in order, beginning with A_0. After we have list-dicolored $A_0 \cup \cdots \cup A_n$, we update the list assignments of each vertex $x \in A_{n+1}$ by removing the colors which appear among both the out-neighbors and the in-neighbors of x.

Theorem 3.2. *Let D be a locally finite Borel digraph on a standard Borel space X. Let Y be Polish, and let $L : X \to [Y]^{<\infty}$ be a Borel list assignment such that $|L(x)| > d^{\max}(x)$ for all $x \in X$. Then D has a Borel L-dicoloring. In particular, D is Borel (degree-plus-one)-list-dicolorable.*

Proof. Since $\tilde{D}$ is locally finite, by Proposition 4.3 in [KST99], there is a countable Borel proper coloring c of $\tilde{D}$. For each $n \in \mathbb{N}$, define $A_n = \{x \in X : c(x) = n\}$. Now let $<_Y$ be a Borel linear ordering on Y, and define $c_0 : A_0 \to Y$ by letting $c_0(x)$ be the $<_Y$-least element of $L(x)$ for each $x \in A_0$.

For each $i \in \mathbb{N}$, write $B_i = \bigcup_{j \le i} A_j$. Let $n > 0$, and assume that, for each $i \le n$, there are a list assignment $L_i : A_i \to [Y]^{<\infty}$ and a Borel function $c_i : B_i \to Y$ such that the following conditions obtain:

- $L_0(x) = L(x)$ for all $x \in A_0$;
- $c_i \restriction B_j = c_j$ for all $j < i$;
- c_i is a Borel L-dicoloring of $D[B_i]$;
- $L_{i+1}(x) = L(x) \setminus \{\alpha \in Y :$ there exist $y, z \in B_i$ such that $y \in N^+(x), z \in N^-(x),$ and $c_i(y) = c_i(z) = \alpha\}$ for all $x \in A_{i+1}$; and
- $|L_{i+1}(x)| > d^{\max}_{D[X \setminus B_i]}(x)$ for all $x \in A_{i+1}$.

We proceed to define a list assignment $L_{n+1} : A_{n+1} \to [Y]^{<\infty}$ and a Borel function $c_{n+1} : B_{n+1} \to Y$ such that:

1. $c_{n+1} \restriction B_i = c_i$ for all $i \le n$;
2. c_{n+1} is a Borel L-dicoloring of $D[B_{n+1}]$;
3. $L_{n+1}(x) = L(x) \setminus \{\alpha \in Y :$ there exist $y, z \in B_n$ such that $y \in N^+(x), z \in N^-(x),$ and $c_n(y) = c_n(z) = \alpha\}$ for all $x \in A_{n+1}$; and
4. $|L_{n+1}(x)| > d^{\max}_{D[X \setminus B_n]}(x)$ for all $x \in A_{n+1}$.

Let $x \in A_{n+1}$, and define $L_{n+1}(x)$ as in (3) above. Note that

$$d^{\max}_{D[X \setminus B_n]}(x) = \max\{d^+(x) - |N^+(x) \cap B_n|, d^-(x) - |N^-(x) \cap B_n|\}$$

and that

$$|L_{n+1}(x)| \ge |L(x)| - \min\{|N^+(x) \cap B_n|, |N^-(x) \cap B_n|\}.$$

Without loss of generality, assume $\max\{d^+(x) - |N^+(x) \cap B_n|, d^-(x) - |N^-(x) \cap B_n|\} = d^+(x) - |N^+(x) \cap B_n|$. Then

$$\begin{aligned} d^+(x) - |N^+(x) \cap B_n| &\le d^{\max}(x) - |N^+(x) \cap B_n| \\ &\le |L(x)| - 1 - |N^+(x) \cap B_n| \\ &\le |L(x)| - 1 - \min\{|N^+(x) \cap B_n|, |N^-(x) \cap B_n|\} \\ &\le |L_{n+1}(x)| - 1. \end{aligned}$$

So, $d^{\max}_{D[X \setminus B_n]}(x) < |L_{n+1}(x)|$. In particular, $|L_{n+1}(x)| \ge 1$. Then for any $x \in B_{n+1}$, we may define $c_{n+1}(x) = c_n(x)$ if $x \notin A_{n+1}$ and $c_{n+1}(x) = \alpha$, where α is the $<_Y$-least element of $L_{n+1}(x)$, if $x \in A_{n+1}$. It is then easy to check that conditions

(1), (3), and (4) above are satisfied by L_{n+1} and c_{n+1}. To see that condition (2) is satisfied, assume for contradiction that there is a c_{n+1}-monochromatic dicycle $C = (x_0, x_1, \ldots, x_k = x_0)$ in B_{n+1}. If $k = 2$, then without loss of generality, assume $c(x_0) > c(x_1)$. Then $x_1 \in N^+(x_0) \cap N^-(x_0) \cap A_{c(x_1)}$, so that $L_{c(x_0)}(x_0)$ does not contain $c_{n+1}(x_1)$, which is a contradiction since $c_{n+1}(x_0) = c_{n+1}(x_1)$. If $k > 2$, then there is $i < k$ such that $c(x_i) > c(x_{i-1})$ and $c(x_i) > c(x_{i+1})$. Then if we set $\alpha = c_{n+1}(x_{i-1}) = c_{n+1}(x_{i+1})$, we have $\alpha \notin L_{c(x_i)}(x_i)$, which is a contradiction since $c_{n+1}(x_i) = \alpha$.

Now define $c' : X \to Y$ by $c' = \bigcup_{n\in\mathbb{N}} c_n$. Then c' is a Borel L-dicoloring of D. □

4. Measurable Brooks's Theorem for Dicolorings

In this section, we prove Theorem 1.6 and Theorem 1.8. The proofs rely heavily on the one-ended spanning forest technique developed by Conley, Marks, and Tucker-Drob in [CMT16]. Recall that, if f is a function on a set X, then f is *one-ended* if there is no sequence $(x_n)_{n\in\mathbb{N}}$ such that, for each $n \in \mathbb{N}$, $f(x_{n+1}) = x_n$. Note that one-ended functions do not have fixed points.

We first show that, if a digraph D of bounded degree admits a Borel one-ended function, then D is Borel degree-list-dicolorable. The proof combines the proof of Theorem 3.2 with the proof of Lemma 3.9 in [CMT16]: First, given a Borel one-ended function f, we separate the graph into layers using the ranks provided by f. Within each layer, we then construct a finite sequence of sets that are independent in the underlying graph and that cover the entire layer. Then we list-dicolor the independent sets in order, removing from the list-assignment of each vertex in the subsequent independent set the colors which appear already among both its out-neighbors and its in-neighbors.

Theorem 4.1. *Let D be a Borel digraph of bounded degree on a standard Borel space X with no isolated vertices, let $B \subseteq X$ be Borel, and let $f : B \to X$ be a one-ended Borel function whose graph is contained in $\tilde{D}$. Let $d \in \mathbb{N}$ be such that $d^{\max}(x) \leq d$ for all $x \in X$. Let Y be Polish, and let $L : X \to [Y]^{<\infty}$ be a Borel list assignment such that $|L(x)| \geq d^{\max}(x)$ for all $x \in B$. Then $D[B]$ has a Borel L-dicoloring. In particular, D is Borel degree-list-dicolorable.*

Proof. For each $n \in \mathbb{N}$, write $f^n[B] = \{x \in X : \text{there exist } x_1, x_2, \ldots, x_n \in B \text{ such that } f(x_1) = x \text{ and } f(x_{i+1}) = x_i \text{ for all } i < n\}$. Let $B_n = B \cap (f^n[B] \setminus f^{n+1}[B])$. Note that, if $n \neq m$, then $B_n \cap B_m = \emptyset$. We claim that $B = \bigcup_{n\in\mathbb{N}} B_n$. Assume for contradiction that there is $x \in B \setminus \bigcup_{n\in\mathbb{N}} B^n$. Then $x \in f^n[B]$ for all $n \in \mathbb{N}$. So the set $f^{-\mathbb{N}}(x) = \{y \in B : \text{for some } n \geq 1, \text{ there are } x_1, x_2, \ldots, x_{n-1} \in B \text{ such that } f(x_1) = x, f(x_{i+1}) = x_i \text{ for all } i < n, \text{ and } f(y) = x_{n-1}\}$ is an infinite, finitely branching tree with root x. By König's lemma, there is an infinite branch $(y_n)_{n\in\mathbb{N}}$ through $f^{-\mathbb{N}}(x)$. Then for each $n \in \mathbb{N}$, $f(y_{n+1}) = y_n$, contradicting that f is one-ended.

We proceed to L-dicolor B one layer at a time. First, we color B_0. Note that $\tilde{D}[B_0]$ is an undirected graph in which each vertex has degree at most $2d$, since $d^{\max}(x) \leq d$ for all $x \in B$. So, by Proposition 4.2 in [KST99], there is a Borel maximal $\tilde{D}$-independent set $B_0^0 \subseteq B_0$. Now, let $0 < i < 2d - 1$, and suppose $B_0^j \subseteq B_0$ has been defined for each $j \leq i$ such that B_0^j is a Borel maximal $\tilde{D}$-independent set in $B_0 \setminus (B_0^0 \cup B_0^1 \cup \cdots \cup B_0^{j-1})$. Then let B_0^{i+1} be a Borel maximal $\tilde{D}$-independent set in $B_0 \setminus (B_0^0 \cup B_0^1 \cup \cdots \cup B_0^i)$. Finally, note that, if $x \in B_0 \setminus (B_0^0 \cup B_0^1 \cup \cdots \cup B_0^{2d-1})$, then $d^{\max}_{D[B_0 \setminus (B_0^0 \cup B_0^1 \cup \cdots \cup B_0^{2d-1})]}(x) = 0$. Then $B_0^{2d} = B_0 \setminus (B_0^0 \cup B_0^1 \cup \cdots \cup B_0^{2d-1})$ is a Borel $\tilde{D}$-independent set. For each $i \leq 2d$, write $B_0^{<i} = \emptyset$ if $i = 0$ and $B_0^{<i} = \bigcup_{j<i} B_0^j$ if $i > 0$, and write $B_0^{\leq i} = \bigcup_{j \leq i} B_0^j$.

Now note that, for each $x \in B_0^0$, $d^{\max}_{D[B_0^0]}(x) = 0$. Since D has no isolated vertices, and since $|L(x)| \geq d^{\max}_D(x)$ for each $x \in X$, it follows that $|L(x)| \geq 1$ for each $x \in B_0^0$. So, by Theorem 3.2, there is a Borel L-dicoloring c_0^0 of $D[B_0^0]$. Now let $1 \leq i < 2d$, and suppose Borel L-dicolorings $c_0^j : B_0^{\leq j} \to Y$ have been defined for all $j \leq i$ so that the following conditions hold:

- If $j' < j$, then $c_0^j \restriction (B_0^{\leq j'}) = c_0^{j'}$; and
- For any $1 \leq j \leq i$ and for any $x \in B_0^j$, if x has an out-neighbor $y \in B_0^{<j}$ and an in-neighbor $z \in B_0^{<j}$ with $\alpha = c_0^{j-1}(y) = c_0^{j-1}(z)$, then $c_0^j(x) \neq \alpha$.

Then for each $x \in B_0^{i+1}$, let

$$L_0^{i+1}(x) = L(x) \setminus \{\alpha \in Y : \text{there are } y, z \in B_0^{\leq i} \text{ such that } c_0^i(y) = c_0^i(z) = \alpha\}.$$

Then, since x has at least one neighbor, namely $f(x)$, not contained in B_0, we have

$$\begin{aligned} |L_0^{i+1}(x)| &\geq |L(x)| - (d^{\max}(x) - 1) \\ &\geq d^{\max}(x) - (d^{\max}(x) - 1) \\ &= 1, \end{aligned}$$

so that $0 = d^{\max}_{D[B_0^{i+1}]}(x) < |L_0^{i+1}(x)|$. Therefore, again by Theorem 3.2, there is a Borel L_0^{i+1}-dicoloring c of $D[B_0^{i+1}]$. Define $c_0^{i+1} : B_0^{\leq i+1} \to Y$ by $c_0^{i+1}(x) = c(x)$ if $x \in B_0^{i+1}$ and $c_0^{i+1}(x) = c_0^i(x)$ otherwise. Finally, define $c_0 = c_0^{2d}$.

We now proceed to dicolor B_1. The procedure is similar to that for dicoloring B_0; we construct $\tilde{D}$-independent sets $B_1^0, B_1^1, \ldots, B_1^{2d}$ as above and color one set at a time to obtain a coloring c_1 of $D[B_1]$. However, in this case, if $x \in B_1^0$, then

$$\begin{aligned} L_1^0(x) = L(x) \setminus \{\alpha \in Y : &\text{there exist } y, z \in B_0 \text{ such that } y \in N^+(x), z \in N^-(x), \\ &\text{and } c_0(y) = c_0(z) = \alpha\}. \end{aligned}$$

Then since x has at least one neighbor, namely $f(x)$, not in B_1, we have $0 = d^{\max}_{D[B_1^0]}(x) < |L_1^0(x)|$, so that $D[B_1^0]$ may be L_1^0-dicolored. The sets $B_2, B_3, \ldots$ are dicolored similarly by functions $c_2, c_3, \ldots$.

Now let $c = \bigcup_{n \in \mathbb{N}} c_n$. Then c is Borel. Assume for contradiction that there is a c-monochromatic cycle $C = (x_0, x_1, \ldots, x_k = x_0)$ in D. If $k = 2$, then let $n \in \mathbb{N}$ be maximal with $C \cap B_n \neq \emptyset$, and let $j \leq 2d$ be maximal with $C \cap B_n^j \neq \emptyset$. Without loss of generality, suppose $x_0 \in B_n^j$. Then $x_1 \notin B_n^j$. So, $c(x_1) \notin L_n^j(x_0)$, a contradiction since $c(x_0) = c(x_1)$. If $k > 2$, then again let $n \in \mathbb{N}$ be maximal with $C \cap B_n \neq \emptyset$, and let $j \leq 2d$ be maximal with $C \cap B_n^j \neq \emptyset$. Then there is $i \leq k$ such that $x_i \in B_n^j$. It follows that $x_{i-1}, x_{i+1} \notin B_n^j$. Then either $x_{i-1} \in B_m$ for some $m < n$, or $x_{i-1} \in B_n^{j'}$ for some $j' < j$, and similarly for x_{i+1}. In any case, it follows that $\alpha = c(x_{i-1}) = c(x_{i+1})$ is not an element of $L_n^j(x_i)$, a contradiction since $c(x_i) = \alpha$. Thus, c is a Borel L-dicoloring of D. □

We will use the following proposition of Conley, Marks, and Tucker-Drob to construct one-ended Borel functions. For a graph G on a set X and for $A \subseteq X$, we use the notation $[A]_G$ to denote the set of all points $x \in X$ such that x has a path through G to some point of A. For a vertex x of G, we write $[x]_G$ for $[\{x\}]_G$. Also, if D is a digraph, then we write $[A]_D$ for $[A]_{\tilde{D}}$.

Proposition 4.2 ([CMT16], Proposition 3.1)**.** *Let G be a locally finite Borel graph on a standard Borel space X, and let $A \subseteq X$ be Borel. Then there is a one-ended Borel function $f : ([A]_G \setminus A) \to [A]_G$ whose graph is contained in G.*

Our next step towards proving Theorem 1.6 is to show that digraphs of bounded degree are Borel degree-list-dicolorable on connected components which contain vertices whose minimum degree is smaller than the maximum degree. To construct the dicoloring, we first reserve a set of small-minimum-degree vertices that is independent in the underlying graph and dicolor the vertices outside of this reserved set using a one-ended function. Then, since each reserved vertex has small minimum degree, there is at least one color in its list of available colors which does not appear among both the out-neighbors and the in-neighbors of the vertex; then the initial dicoloring can be extended to this vertex. The proof resembles the first part of the proof of Theorem 1.2 in [CMT16].

Proposition 4.3. *Let D be a Borel digraph of bounded degree on a standard Borel space X, and let $B = \{x \in X : d^{\min}(x) < d^{\max}(x)\}$. Then $D[[B]_D]$ is Borel degree-list-dicolorable.*

Proof. Let Y be Polish, and let $L : X \to [Y]^{<\infty}$ be a list assignment such that $|L(x)| \geq d^{\max}(x)$ for each $x \in X$. Since D is of bounded degree, the underlying graph $\tilde{D}$ is also of bounded degree. Therefore, by Proposition 4.2 in [KST99], there is a Borel maximal $\tilde{D}$-independent set $B' \subseteq B$. Since $[B']_D = [B]_D$, by Proposition 4.2, there is a one-ended Borel function $f : ([B]_D \setminus B') \to [B]_D$ whose graph is contained in $\tilde{D}$. Then by Theorem 4.1, $D[[B]_D \setminus B']$ has a Borel L-dicoloring.

Now we extend c to a function $c' : [B]_D \to Y$ by letting $c'(x) = c(x)$ if $x \notin B'$; if $x \in B'$, then let $<_Y$ be a Borel linear ordering of Y, and define $c'(x)$ to be the $<_Y$-least color $\alpha \in L(x)$ such that there are no $y \in N^+(x) \cap ([B]_D \setminus B')$ and $z \in N^-(x) \cap ([B]_D \setminus B')$ with $c(y) = c(z) = \alpha$. Such an α exists since $d^{\min}(x) < d^{\max}(x)$, so that at most $d^{\max}(x) - 1$ colors from $L(x)$ appear in $c[N^+(x)] \cap c[N^-(x)]$. Then c' is a Borel L-dicoloring of $D[[B]_D]$. □

From now on, we say that D is *Eulerian* if, for each vertex, its in-degree and out-degree are the same. So, the proposition above demonstrates that, if D is a bounded-degree Borel digraph that is not Eulerian, then D is Borel degree-list-dicolorable.

As a corollary to the proof of Proposition 4.3, we obtain the following.

Corollary 4.4. *Let D be a Borel digraph on a standard Borel space X, and let d be such that $d^{\max}(x) \leq d$ for all $x \in X$. Let $B = \{x \in X : d^{\min}(x) < d\}$. Then $D[[B]_D]$ is Borel d-dicolorable.*

Proof. Take B' and c as in the proof of the previous proposition, and let $L(x) = \{0, 1, \dots, d-1\}$ for all $x \in X$. Then extend c to a function c' on X by setting $c'(x) = c(x)$ for all $x \notin B'$; for each $x \in B'$, let $c'(x) = \alpha$, where α is the least color such that α does not belong to both $N^+(x) \cap ([B]_D \setminus B')$ and $N^-(x) \cap ([B]_D \setminus B')$. Such an α exists since $d^{\min}(x) < d$. □

Next we work towards proving Theorem 1.8, which will be used in the proof of Theorem 1.6. In particular, we show that digraphs of bounded degree are Borel degree-list-dicolorable on connected components which are not Gallai trees. The proof is similar to the proof of Theorem 4.1 in [CMT16]. The argument again involves reserving a set B'' of points that belong to biconnected sets which do not induce dicycles, odd symmetric cycles, or complete symmetric digraphs and which are sufficiently well-separated in the underlying graph. Once a degree-list-dicoloring a of the points outside B'' has been constructed, the points of B'' will be colored in a two-step procedure: First, the values of a are changed on certain neighborhoods of the points in B''; then, the dicoloring that results from changing a on these neighborhoods will be extended until all points are colored. The separation between the points of B'' will guarantee that the alterations made to a do not interfere with one another.

The biconnected sets to which the points of B'' will belong will be required to have the following form.

Definition 4.5. Let D be a digraph on a set X. Let $G = (x_0, x_1, \ldots, x_k = x_0) \subseteq X$ be a cycle in $\tilde{D}$, so that, for each $i < k$, x_i is $\tilde{D}$-adjacent to x_{i+1} and, for all $i, j < k$, $x_i \neq x_j$ if $i \neq j$. Then G is a **good cycle** if the following two conditions hold:

1. There is some orientation of the arcs in G that does not yield a dicycle; and
2. $D[G]$ is not an odd symmetric cycle or a complete symmetric digraph.

The following proposition can be deduced from the proof of Lemma 2.1 in [HM11]. It shows that any "good" biconnected set of size at least 3 contains a good cycle.

Proposition 4.6 ([HM11])**.** *Let D be an Eulerian digraph on a set X, and let $M \subseteq X$ be a finite biconnected set in D such that $|M| \geq 3$ and M does not induce a dicycle, an odd symmetric cycle, or a complete symmetric digraph. Then there is a good cycle contained in M.*

Consider an Eulerian digraph D. Let L be a list assignment such that $|L(x)| \geq d^{\max}(x)$ for all vertices x, and assume that all vertices of D except one vertex x_0 have been L-dicolored according to a function c. If $|L(x_0)| > d$, or if $|L(x_0)| = d$ and the number of colors in $L(x_0)$ that appear among both the in-neighbors and the out-neighbors of x_0 is less than d, then the coloring c may be extended to x_0 by defining $c(x_0)$ to be the least color in $L(x_0)$ which does not appear among both the out-neighbors and the in-neighbors of x_0. Otherwise, the following proposition of Harutyunyan and Mohar shows that, by uncoloring any neighbor y of x_0 and then coloring x_0 with the color that y previously had, we obtain a new L-dicoloring c' of $D[X \setminus \{y\}]$.

Proposition 4.7 ([HM11], Lemma 2.2)**.** *Let D be an Eulerian digraph on a set X, and let L be a list assignment such that $|L(x)| \geq d^{\max}(x)$ for all $x \in X$. Let $x_0 \in X$, and suppose c is an L-dicoloring of $D[X \setminus \{x_0\}]$ such that, for each $\alpha \in L(x_0)$, x_0 has both an out-neighbor and an in-neighbor of color α. Let y be a neighbor of x_0, and define c' on $X \setminus \{y\}$ by $c'(x_0) = c(y)$ and $c'(z) = c(z)$ if $z \neq x_0$. Then c' is an L-dicoloring of $D[X \setminus \{y\}]$.*

The final result that we need before proving Theorem 1.8 can be deduced from Lemmas 2.4 and 2.5 in [HM11]. It shows that, if G is a good cycle and all vertices of D except one vertex in G have been L-dicolored, then by repeatedly uncoloring and then coloring pairwise adjacent vertices in G as in the statement of Proposition 4.7, we reach a vertex $y \in G$ for which some element of $L(y)$ does not appear among both the out-neighbors and the in-neighbors of y. At this stage, the L-dicoloring may be extended to y.

Proposition 4.8 ([HM11]). *Let D be an Eulerian digraph on a set X, and let L be a list assignment for D such that $|L(x)| \geq d^{\max}(x)$ for all $x \in X$. Let $G \subseteq X$ be a good cycle. Let $x \in G$, and suppose that there is an L-dicoloring a of $D[X \setminus \{x\}]$. Then there are a sequence $(x = x_0, x_1, \ldots, x_k)$ of vertices in G and a sequence $(a = a_0, a_1, \ldots, a_k)$ of functions such that, for each $i < k$:*

(1) *x_i is $\tilde{D}$-adjacent to x_{i+1};*
(2) *a_i is an L-dicoloring of $D[X \setminus \{x_i\}]$, and a_k is an L-dicoloring of $D[X \setminus \{x_k\}]$;*
(3) *For each $\alpha \in L(x_i)$, $\alpha \in a_i[N^-(x_i)] \cap a_i[N^+(x_i)]$;*
(4) *a_{i+1} is obtained from a_i by uncoloring x_{i+1} and coloring x_i with $a_i(x_{i+1})$ (i.e., through an application of Proposition 4.7); and*
(5) *There is some color $\alpha \in L(x_k)$ such that either $\alpha \notin a_k[N^-(x_k)]$ or $\alpha \notin a_k[N^+(x_k)]$.*

Finally, recall that the *boundary* of a set S in the digraph $D = (X, A)$, denoted ∂S, is the set $\partial S = \{x \in X : x \notin S$ but there exists $y \in S$ such that $x \, \tilde{D} \, y\}$.

Now we prove Theorem 1.8. We restate the theorem here for convenience.

Theorem 4.9. *Let D be a Borel digraph of bounded degree on a standard Borel space X. Let L be a Borel list assignment for D such that $|L(x)| \geq d^{\max}(x)$ for all $x \in X$, and let $B = \{x \in X : D[[x]_D]$ is not a Gallai tree$\}$. Then $D[B]$ has a Borel L-dicoloring. That is, $D[B]$ is Borel degree-list-dicolorable.*

Proof. Note that, by Theorem 4.3, we may assume that $d^{\max}(x) = d^{\min}(x)$ for all $x \in X$, that is, that D is Eulerian.

Throughout the proof, fix $d \in \mathbb{N}$ such that $d^{\max}(x) \leq d$ for all $x \in X$. Also, we shall call a biconnected set S *bad* if the sub-digraph that S induces is a dicycle, an odd symmetric cycle, or a complete symmetric digraph. For technical reasons, we assume first that each block of $D[B]$ which is not bad has cardinality at least 3. We shall explain at the end of the proof that there is a Borel L-dicoloring of the connected components of $D[B]$ which have non-digon blocks of cardinality 2.

Let

$$[E_D]^{<\infty} = \{S \in [X]^{<\infty} : S \text{ is contained in a connected component of } D\},$$

and let

$$A = \{S \in [E_D]^{<\infty} : \text{each connected component of } D[S] \text{ contains a block of cardinality at least 3 which is not bad}\}.$$

Define G_I to be the intersection graph on $[E_D]^{<\infty}$, so that $S, T \in [E_D]^{<\infty}$ are G_I-adjacent if and only if $S \neq T$ and $S \cap T \neq \emptyset$. Then by Proposition 2 of [CM16], there is a countable Borel proper coloring c_I of G_I. Let now

$$A' = \{S \in A : c_I(S \cup \partial S) \leq c_I(T \cup \partial T) \text{ for all } T \in A \text{ in the same } D\text{-component as } S\},$$

and define $B' = \bigcup A'$.

We next prove several claims about B'. First, we need the following.

Claim 1. Let $x \in X$. Then $D[[x]_D]$ is not a Gallai tree if and only if there is a finite connected set $T \subseteq [x]_D$ such that the induced sub-digraph $D[T]$ contains a block of cardinality at least 3 which is not bad.

Proof of Claim 1. ($\Rightarrow$) Suppose $D[[x]_D]$ is not a Gallai tree. Then there is a block in $[x]_D$ that does not induce a dicycle, an odd symmetric cycle, or a complete symmetric digraph. If there is a finite such block M in $[x]_D$, then since M has cardinality at least 3 by the assumption at the beginning of the proof, we may take $T = M$ in the statement of the claim.

Otherwise, there is an infinite block M in $[x]_D$. Now we take inspiration from the proof of Lemma 2.1 in [HM11]. Let $y, z \in M$ be such that $d(y, z) \geq d + 2$; such y, z exist because M is an infinite connected subset of a locally finite graph. Since M is biconnected, it follows from a classical result of Whitney ([Whi32a], [Whi32b]) that there are two internally-vertex-disjoint paths P_0, P_1 from y to z. Note that each of P_0, P_1 has length at least $d+2$. Again since the $(d+2)$-neighborhood of $P_0 \cup P_1$ is a finite set in a locally finite graph, there is some $v \in M$ such that $d(v, P_0 \cup P_1) \geq d+2$. Since M is connected, there is a path $Q = (y = q_0, q_1, \ldots, q_k = v)$ through M from y to v.

Case 1: Q contains some point of $P_0 \cup P_1$ besides y. Let $i < k$ be maximal such that $q_i \neq y$ and $q_i \in P_0 \cup P_1$, and write $y' = q_i$. Since M is biconnected, there is a path $R = (y = r_0, r_1, \ldots, r_l = v)$ through M from y to v that is internally-vertex-disjoint from Q. Let $j < l$ be maximal such that $r_j \in P_0 \cup P_1$, and write $y'' = r_j$. Then y', y'' are distinct points of $P_0 \cup P_1$, and there are three internally-vertex-disjoint paths from y' to y'': These are the two paths S_0, S_1 arising from the fact that y', y'' are distinct points of the cycle $P_0 \cup P_1$, together with the concatenation S_2 of the sub-path of Q from y' to v and the sub-path of R from v to y''. Notice that the length of S_2 and the length of at least one of S_0, S_1 are at least $d + 2$. So, two of the three paths S_0, S_1, S_2 form a cycle C of even length with cardinality at least $d + 2$. Since each vertex of D has maximum degree at most d, the induced sub-digraph $D[C]$ is not a complete symmetric digraph. So, if $D[C]$ is not a dicycle, then we may take $T = C$ in the statement of Claim 1. If $D[C]$ is a dicycle, then we may take $T = S_0 \cup S_1 \cup S_2$.

Case 2: The only point of $P_0 \cup P_1$ in Q is y. Again since M is connected, there is a path R through M from z to v. If R contains some point of $P_0 \cup P_1$ besides z, then we may proceed as in Case 1 by replacing y with z and Q with R. Otherwise, there are three internally-vertex-disjoint paths P_0, P_1, and $R \cup Q$ (with repetitions removed) between y and z. Note that each of these paths has length at least $d + 2$. Then we may again proceed as in Case 1 by replacing S_0, S_1, S_2 with $P_0, P_1, R \cup Q$.

($\Leftarrow$) Suppose there is a finite connected set $T \subseteq [x]_D$ such that $D[T]$ contains a block M of size at least 3 which is not bad. If there is an infinite block in $[x]_D$ which contains M, then clearly $D[[x]_D]$ is not a Gallai tree, and the proof is complete. If the only blocks in $[x]_D$ which contain M are finite, then since M has cardinality at least 3, there is no block in D containing M that induces a dicycle, an odd symmetric cycle, or a complete symmetric digraph. So $D[[x]_D]$ is again not a Gallai tree. $\square$

Now we claim $B' \subseteq B$. Let $x \in B'$. Then there is $S \in A'$ such that $x \in S$. Because each connected component of $D[S]$ contains a block of size at least 3 which is not bad, it follows from Claim 1 that $D[[x]_D]$ is not a Gallai tree. So $x \in B$. Furthermore, Claim 1 implies that each connected component of $D[B]$ meets B'.

Next, we show that each connected component of $D[B']$ is finite. Let $x \in B'$; then there is some $S \in A'$ such that $x \in S$. We claim $[x]_{D[B']} \subseteq (S \cup \partial S)$, which is a finite set. Assume for contradiction that there is some $y \in B'$ such that $y \notin S \cup \partial S$, but there is a path $(x = x_0, x_1, \ldots, x_k = y)$ through $\tilde{D}[B']$ from x to y. Since $y \notin S \cup \partial S$ and $x \in S$, there is $i < k$ such that $x_i \in \partial S$ and $x_{i+1} \notin S \cup \partial S$. Let $T \in A'$ be such that $x_{i+1} \in T$. Then $(S \cup \partial S) \cap (T \cup \partial T)$ contains x_i, so that $c_I(S \cup \partial S) \neq c_I(T \cup \partial T)$. This is a contradiction since $S, T \in A'$ belong to the same connected component of D.

Note now that each connected component C of $D[B']$ contains a biconnected set M of size at least 3 that is not bad. Let $<_X$ be a Borel linear ordering of X; then we define M_C to be the lexicographically least such M, assuming without loss of generality that the elements of any finite subset of X are listed in increasing order according to $<_X$. Further, we may assume that, if C, C' are distinct connected components of $D[B']$, then M_C and $M_{C'}$ are such that $(M_C \cup \partial M_C) \cap (M_{C'} \cup \partial M_{C'}) = \emptyset$; this is because, if $S \subseteq C$ and $T \subseteq C'$ are elements of A', then $S \cup \partial S \neq T \cup \partial T$, so that $(S \cup \partial S) \cap (T \cup \partial T) = \emptyset$.

For each connected component C of $D[B']$, by Proposition 4.6, there is a good cycle G in M_C; let G_C be the lexicographically least such cycle according to $<_X$, and let x_C be the $<_X$-least element of G_C. Finally, set $B'' = \{x \in X :$ there is a connected component C of $D[B']$ such that $x = x_C\}$. Then B'' is Borel, and $[B'']_D = [B']_D = B$.

So, by Proposition 4.2, there is a one-ended Borel function from $(B \setminus B'')$ to B. Then by Theorem 4.1, there is a Borel L-dicoloring a of $D[B \setminus B'']$.

We proceed to define a Borel L-dicoloring a' of $D[B]$. For each $x \in B''$, let C be the connected component of $D[B']$ witnessing $x \in B''$, and let $(x = x_0, x_1, \ldots, x_k), (a = a_0, a_1, \ldots, a_k)$ be as in Proposition 4.8; this proposition may be applied since G_C is a good cycle and $x = x_C \in G_C$. Then the L-dicoloring a_k may be extended to x_k, since there is some color $\alpha \in L(x_k)$ such that either $\alpha \notin a_k[N^-(x_k)]$ or $\alpha \notin a_k[N^+(x_k)]$; write a_C for this extension of a_k. Then define $a' : B \to Y$ by $a'(x) = a_C(x)$ if $x \in G_C$ and $a'(x) = a(x)$ if $x \notin G_C$ for any C.

We claim that a' is a dicoloring of $D[B]$. It is enough to show the following.

Claim 2. Let $x, y \in B''$ be distinct. Let c be a dicoloring of $D[X \setminus \{x, y\}]$ such that, for each $\alpha \in L(x)$, $\alpha \in c[N^-(x)] \cap c[N^+(x)]$, and for each $\beta \in L(y)$, $\beta \in c[N^-(y)] \cap c[N^+(y)]$. Let x' be a neighbor of x, and let y' be a neighbor of y. Then the function c' defined on $X \setminus \{x', y'\}$ by $c'(z) = c(z)$ if $z \neq x, y$, $c'(x) = c(x')$, and $c'(y) = c(y')$ is an L-dicoloring of $D[X \setminus \{x', y'\}]$.

Proof of Claim 2. Note first that, since $x, y \in B''$ are distinct, then if C, C' are the connected components of B' witnessing that $x, y \in B''$, respectively, then $(G_C \cup \partial G_C) \cap (G_{C'} \cup \partial G_{C'}) = \emptyset$. Now assume for contradiction that there is a c'-monochromatic dicycle K. Then K cannot contain x' or y', since these vertices are uncolored. Further, K must contain either x or y; otherwise, there is a c-monochromatic dicycle, contradicting that c is a dicoloring. Suppose $x \in K$. Then $c'(x) = c(x')$. Since, for each $\alpha \in L(x)$, $\alpha \in c[N^-(x)] \cap c[N^+(x)]$, and since $(G_C \cup \partial G_C) \cap (G_{C'} \cup \partial G_{C'}) = \emptyset$ so that y and y' are not neighbors of x, it follows that there is exactly one neighbor z of x such that $c'(x) = c'(z)$. Therefore, any dicycle passing through x either contains points of different colors or contains an uncolored point. So, no dicycle containing x is monochromatic. A similar argument shows that no dicycle containing y is monochromatic. □

This completes the proof in the case where each block of $D[B]$ that is not bad has cardinality at least 3. For the other case, let $B_0 = \{x \in X : [x]_D$ has a non-digon block of cardinality $2\}$. As above, take a Borel set A' of non-digon blocks in B_0 such that, if M, M' are elements of A' with $M \cup \partial M \neq M' \cup \partial M'$, then $(M \cup \partial M) \cap (M' \cup \partial M') = \emptyset$ and such that A' meets each connected component of $D[B_0]$. Then set $B' = \bigcup A'$. Use a Borel linear ordering to select from each connected component C of B' a point x_C that belongs to a non-digon block of cardinality 2, and collect these points x_C into a set B''. Then use Theorem 4.1 to produce a Borel L-dicoloring a of $D[B_0 \setminus B'']$. To color the points of B'', note that, if $x \in B''$, then there is some neighbor y of x such that $\{x, y\}$ is a maximal biconnected set that is not a digon. In particular, no dicycle contains both x and y. So, if $L(x)$ contains $a(y)$, then a may be extended to x by setting $a(x) = a(y)$. Otherwise, there is some color in $L(x)$ which does not appear among both the out-neighbors and the in-neighbors of x, and so again a can be extended to x by setting $a(x)$ to be the least such color. □

Now we have nearly all the tools required to prove Theorem 1.6. We next need a result of [CMT16] which shows that, modulo a null set or a meager set, an undirected acyclic graph in which no connected component has 0 or 2 ends admits a one-ended Borel function. Recall that, if G is a locally finite graph on a set X, then a set $B \subseteq X$ is *G-invariant* if, whenever $x \in B$ and there is a path from x to y through G, then $y \in B$. If D is a digraph on X, then we say $B \subseteq X$ is *D-invariant* if B is $\tilde{D}$-invariant. A *ray* in G is an infinite sequence $(x_n)_{n\in\mathbb{N}}$ of pairwise-adjacent vertices in G such that $x_n \neq x_m$ whenever $n \neq m$. Two rays r_0, r_1 in G are *end-equivalent* if, whenever $S \subseteq X$ is finite, r_0 and r_1 eventually lie in the same connected component of $G[X \setminus S]$. End-equivalence is an equivalence relation on the set of rays; the equivalence classes are called *ends*.

Theorem 4.10 ([CMT16], Theorem 1.5)**.** *Let G be a locally finite acyclic graph on a standard Borel space X. Assume no connected component of G has either* 0 *or* 2 *ends.*

(1) For any Borel probability measure μ on X, there are a μ-conull, G-invariant Borel set B and a one-ended Borel function $f : B \to X$ whose graph is contained in G.
(2) For any Polish topology τ compatible with the Borel structure on X, there are a τ-comeager, G-invariant Borel set B and a one-ended Borel function $f : B \to X$ whose graph is contained in G.

The proof of the following theorem resembles the proof of Theorem 4.2 in [CMT16].

Theorem 4.11. *Let D be a Borel digraph of bounded degree on a standard Borel space X. Assume that D has no finite connected components that are Gallai trees and no infinite connected components that are* 2*-ended Gallai trees. Then:*

(1) For any Borel probability measure μ on X, there is a μ-conull, D-invariant Borel set B such that $D[B]$ is Borel degree-list-dicolorable.
(2) For any Polish topology τ compatible with the Borel structure on X, there is a τ-comeager, D-invariant Borel set B such that $D[B]$ is Borel degree-list-dicolorable.

Proof. We prove (1); the proof of (2) proceeds in the same way.

Let L be a list assignment such that $|L(x)| \geq d^{\max}(x)$ for all $x \in X$. By the assumptions in the theorem statement together with Theorem 4.9, we may assume without loss of generality that each connected component of D is an infinite Gallai tree that does not have 2 ends. Then each block in D induces a finite dicycle, a finite odd symmetric cycle, or a finite complete symmetric digraph. So, in the underlying graph $\tilde{D}$, each block induces either a cyclic graph or a complete graph.

Now let E denote the set of edges of $\tilde{D}$. Note that each element of E is contained in a unique block. We now create a subset $E' \subseteq E$ as follows. Let $<_X$ be a Borel linear ordering of X, and let $<_E$ be a Borel linear ordering of E. From each block of $\tilde{D}$ that induces a cyclic graph, remove the $<_E$-least edge; from each block M of $\tilde{D}$ that induces a complete graph of cardinality at least 3, write $M = \{x_0, x_1, \ldots, x_k\}$ in ascending order according to $<_X$, and then delete all edges of M except $\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_k, x_0\}$ before also deleting the $<_E$-least edge in the list $\{x_0, x_1\}, \{x_1, x_2\}, \ldots, \{x_k, x_0\}$. Now define a sub-digraph D' of D whose vertex set is X and whose arc set is A', where $(x, y) \in A'$ if and only if $(x, y) \in A$ and $\{x, y\} \in E'$.

We prove several claims about D'. Note first that, if x, y are adjacent in D, then there is a path from x to y through $\widetilde{D'}$; indeed, the edge between x and y in the underlying graph $\tilde{D}$ determines a unique block M. If M has cardinality 2, then the edge $\{x, y\}$ belongs to E'. Otherwise, if $\tilde{D}[M]$ is a cycle, then there is a path through M from x to y which does not contain the edge $\{x, y\}$; if $\tilde{D}[M]$ is a complete graph, then $\widetilde{D'}[M]$ is a cycle missing just one edge, so that again there is a path through M from x to y which does not contain the edge $\{x, y\}$. This implies that no connected component of D' is 0-ended, since no connected component of D is 0-ended. This also implies that, if $B \subseteq X$ is D'-invariant, then B is also D-invariant.

Next, to see that D' is acyclic, note that any cycle C in $\tilde{D}$ induces either a cyclic graph or a complete graph. If C induces a cyclic graph, then some edge of C is deleted in passing from D to D'. If C induces a complete graph, then let M be the maximal complete graph containing C. Then at least one of the edges of M that is removed in passing to D' is an edge of C.

Finally, to see that no connected component of D' is 2-ended, note first that the number of ends in a connected component of D is less than or equal to the number of ends in the corresponding connected component of D'. Suppose now that there are two rays r, r' in the same connected component of D' such that, for some finite set $S \subseteq X$, $r \setminus S$ and $r' \setminus S$ do not eventually lie in the same connected component of $D'[X \setminus S]$. Let $T = S \cup \bigcup \{M \in [E_D]^{<\infty} : M \text{ is a block containing a point of } S\}$. Then it is easy to prove that $r \setminus T$ and $r' \setminus T$ do not eventually lie in the same connected component of $D[X \setminus T]$. So, since no connected component of D is 2-ended, no connected component of D' is 2-ended.

It now follows from Theorem 4.10 that there are a μ-conull, D'-invariant Borel set B and a one-ended Borel function $f : B \to X$ whose graph is contained in $\tilde{D}'$. Note that B is D-invariant by the claim above and that the graph of f is contained in $\tilde{D}$. By Theorem 4.1, $D[B]$ has a Borel L-dicoloring. □

Finally, we prove Theorem 1.6. We restate the theorem here for convenience.

Theorem 4.12. *Let D be a Borel digraph on a standard Borel space X. Suppose there is $d \geq 3$ such that $d^{\max}(x) \leq d$ for all $x \in X$, and assume D does not contain the complete symmetric digraph on $d + 1$ vertices. Then:*

(1) For any Borel probability measure μ on X, there is a μ-measurable d-dicoloring of D.

(2) *For any Polish topology τ compatible with the Borel structure on X, there is a τ-Baire-measurable d-dicoloring of D.*

Proof. By Corollary 4.4, we may assume that $d^{\max}(x) = d^{\min}(x) = d$ for all $x \in X$. Note that, if all the vertices in a finite Gallai tree have out-degree and in-degree equal to d, then the Gallai tree is the complete symmetric digraph on d vertices. Also, if all the vertices in an infinite 2-ended Gallai tree have out-degree and in-degree equal to d, then either $d = 1$ and the tree is a one-directional bi-infinite line, or $d = 2$ and the tree is a symmetric bi-infinite line. Since $d \geq 3$, it follows from Theorem 4.11 that D has a (μ- or τ-)measurable d-dicoloring. □

5. Future Directions

Here we outline a potential connection between descriptive digraph combinatorics and LOCAL algorithms. Bernshteyn's seminal paper [Ber23] reveals a deep connection between descriptive graph combinatorics and *LOCAL coloring algorithms*, which are efficient distributed algorithms for graph coloring. In particular, deterministic LOCAL coloring algorithms can be used to produce Borel graph colorings ([Ber23], Theorem 2.10), and randomized LOCAL coloring algorithms can be used to produce (μ- or Baire-)measurable graph colorings ([Ber23], Theorem 2.14).

The main tool in the proof of the latter result is a measurable version of the *Lovász local lemma*, an important theorem of probability theory. Let $p < 1$, and consider a set $\mathcal{S}$ of events, each of which has probability at most p. Suppose that each event in $\mathcal{S}$ depends on only a small number d of other events in $\mathcal{S}$. If p and d satisfy a certain relationship (in particular, if $ep(d+1) \leq 1$), then the local lemma ensures that the probability that none of the events in $\mathcal{S}$ occurs is nonzero. In the graph coloring context, we may, loosely speaking, take $\mathcal{S}$ to be the set of events in which some vertex in a graph shares a color with one of its neighbors; for a graph of bounded degree, each such event has probability bounded away from 1. Furthermore, due to the local nature of the problem of producing a proper coloring, each event in $\mathcal{S}$ depends on only a small number of other events in $\mathcal{S}$. Bernshteyn's measurable local lemma then ensures the existence of a measurable coloring in which none of the events obtains, that is, a measurable proper coloring.

We would like to explore whether there is a digraph version of Bernshteyn's result that randomized LOCAL algorithms yield measurable graph colorings. One strategy would be to attempt to apply the measurable local lemma to obtain measurable dicolorings. However, unlike the problem of producing a proper coloring, the problem of producing a dicoloring is not a local problem; directed cycles can be arbitrarily long. So, a direct application of the measurable local lemma is not possible. Instead, we ask the following question.

Question 5.1. *Is there an extension of Bernshteyn's measurable local lemma that is amenable to digraph combinatorics?*

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