MEASURABLE BROOKS’S THEOREM FOR DIRECTED GRAPHS

CECELIA HIGGINS

Abstract. We prove a definable version of Brooks’s theorem for directed graphs. In particular, we show that, if \( D \) is a Borel directed graph on a standard Borel space \( X \) in which the maximum degree of each vertex is at most \( d \geq 3 \), then unless \( D \) contains the complete symmetric directed graph on \( d + 1 \) vertices, \( D \) admits a \( \mu \)-measurable dicoloring with \( d \) colors with respect to any Borel probability measure \( \mu \) on \( X \), and \( D \) admits a \( \tau \)-Baire-measurable dicoloring with \( d \) colors with respect to any Polish topology \( \tau \) compatible with the topology on \( X \).

1. Introduction

A classical graph theory result states that, for a finite undirected graph \( G \) in which each vertex has degree at most \( d \), the chromatic number of \( G \) is at most \( d + 1 \). It is easy to see that this upper bound is sharp. For \( d = 2 \), if \( G \) is an odd cycle, then each vertex of \( G \) has degree 2, but the chromatic number of \( G \) is 3; for \( d \geq 3 \), if \( G \) is the complete graph on \( d + 1 \) vertices, then each vertex of \( G \) has degree \( d \), but the chromatic number of \( G \) is \( d + 1 \).

However, these obvious obstructions to a smaller upper bound are, in a precise sense, the only obstructions. The following 1941 theorem of Brooks characterizes the graphs of bounded degree degree \( d \) which are \( d \)-colorable as the graphs which do not contain odd cycles or complete graphs.

Theorem 1.1 ([Bro41]). Let \( G \) be a finite undirected graph in which each vertex has degree at most \( d \). If \( d = 2 \), assume \( G \) has no odd cycles; if \( d \geq 3 \), assume \( G \) does not contain the complete graph on \( d + 1 \) vertices. Then \( G \) has a proper coloring with \( d \) colors.

Brooks’s proof of this theorem uses classical techniques. Results of Marks ([Mar16]) demonstrate that Brooks’s theorem fails in the Borel context. However, in 2016, Conley, Marks, and Tucker-Drob gave a measurable version of Brooks’s theorem.

Theorem 1.2 ([CMT16], Theorem 1.2). Let \( G \) be an undirected Borel graph on a standard Borel space \( X \) in which each vertex has degree at most \( d \geq 3 \). Assume \( G \) does not contain the complete graph on \( d + 1 \) vertices. Then:

1. For any Borel probability measure \( \mu \) on \( X \), there is a \( \mu \)-measurable proper coloring of \( G \) with \( d \) colors.

2. For any Polish topology \( \tau \) compatible with the Borel structure on \( X \), there is a \( \tau \)-Baire-measurable proper coloring of \( G \) with \( d \) colors.

Date: February 16, 2024.
In this note, we show that the above theorem has an analogue in the setting of directed graphs, in which each edge between vertices is viewed as having an orientation.

**Definition 1.3.** A **directed graph** (or **digraph**) $D$ is a pair $D = (X, A)$, where $X$ is a set, called the **vertex set**, and $A \subseteq X^2$ is an irreflexive relation. The elements of $A$ are called **arcs**.

The appropriate generalization of the notion of proper coloring for directed graphs was introduced first by Neumann-Lara (NL82) in 1982 and again by Mohar (Moh03) in 2003.

**Definition 1.4.** A **directed cycle** (or **dicycle**) in a digraph $D$ is a set $C = (x_0, x_1, \ldots, x_k)$ of vertices in $D$ such that $x_0 = x_k$ and there is an arc from $x_i$ to $x_{i+1}$ for each $i \in \mathbb{N}$ such that $0 \leq i < k$. If $d \in \mathbb{N}$, then a $d$-**dicoloring** of $D$ is a function $c : X \to \{1, 2, \ldots, d\}$ such that, for any dicycle $C$ in $D$, there are points $x, x' \in C$ such that $c(x) \neq c(x')$. The **dichromatic number** of $D$, denoted $\chi^\rightarrow(D)$, is the least $n \in \mathbb{N}$ such that there is a dicoloring of $D$ with $n$ colors.

The following theorem, which was first stated by Mohar in 2010 and later proved in full by Harutyunyan and Mohar in 2011, characterizes the obstacles to dicoloring with a small number of colors as the directed-graph analogues of odd cycles and complete graphs, along with directed cycles.

**Theorem 1.5** (Moh10, Theorem 2.3; HMI11, Theorem 2.1). Let $D$ be a finite directed graph in which each vertex has maximum degree at most $d$. If $d = 1$, assume $D$ has no directed cycles; if $d = 2$, assume $D$ has no symmetric odd cycles; and if $d \geq 3$, assume $D$ does not contain the complete symmetric digraph on $d + 1$ vertices. Then $D$ has a dicoloring with $d$ colors.

In this note, we give a definable version of this theorem. In particular, we show the following.

**Theorem 1.6.** Let $D$ be a Borel digraph on a standard Borel space $X$. Suppose there is $d \geq 3$ such that $d^{\max}(x) \leq d$ for all $x \in X$, and assume $D$ does not contain the complete symmetric graph on $d + 1$ vertices. Then:

1. For any Borel probability measure $\mu$ on $X$, there is a $\mu$-measurable dicoloring of $D$ with $d$ colors.
2. For any Polish topology $\tau$ compatible with the Borel structure on $X$, there is a $\tau$-Baire-measurable dicoloring of $D$ with $d$ colors.

In future work, we would like to better understand the implications of this result for definable digraph combinatorics more generally. In particular, although dicoloring is not a local problem in general, we would like to study possible connections between definable digraph combinatorics and **LOCAL** algorithms; this may lead to interesting extensions of Bernshteyn’s work (Ber23).

2. Preliminaries

We begin by recalling the basic terminology and notation for directed graphs. Throughout the rest of this section, let $D$ be a directed graph on a set $X$.

**Definition 2.1.** Let $x \in X$. 
(1) An element $y \in X$ is an out-neighbor of $x$ if there is an arc from $x$ to $y$.

The set of out-neighbors of $x$ is denoted $N^+(x)$.

(2) An element $y \in X$ is an in-neighbor of $x$ if there is an arc from $y$ to $x$.

The set of in-neighbors of $x$ is denoted $N^-(x)$.

(3) The out-degree of $x$, denoted $d^+(x)$, is defined by $d^+(x) = |N^+(x)|$.

(4) The in-degree of $x$, denoted $d^-(x)$, is defined by $d^-(x) = |N^-(x)|$.

(5) The maximum degree of $x$, denoted $d_{\text{max}}(x)$, is defined by $d_{\text{max}}(x) = \max\{d^+(x), d^-(x)\}$.

(6) The minimum degree of $x$, denoted $d_{\text{min}}(x)$, is defined by $d_{\text{min}}(x) = \min\{d^+(x), d^-(x)\}$.

(7) The maximum side of $x$, denoted $N_{\text{max}}(x)$, is defined by $N_{\text{max}}(x) = N^+(x)$ if $|N^+(x)| \geq |N^-(x)|$ and $N_{\text{max}}(x) = N^-(x)$ otherwise.

(8) The minimum side of $x$, denoted $N_{\text{min}}(x)$, is defined by $N_{\text{min}}(x) = N^+(x)$ if $|N^+(x)| \leq |N^-(x)|$ and $N_{\text{min}}(x) = N^-(x)$ otherwise.

Note that, in the above setting, it is possible that $N^+(x) \cap N^-(x) \neq \emptyset$.

Now we define the main obstacles to dicoloring.

**Definition 2.2.** The digraph $D$ is a symmetric cycle if $D$ has the form $(x_0, x_1, \ldots, x_k = x_0)$ such that, for each $i < k$, there is both an arc from $x_i$ to $x_{i+1}$ and an arc from $x_{i+1}$ to $x_i$. The graph $D$ is a complete symmetric digraph if, whenever $x, y \in X$ are distinct, there are an arc from $x$ to $y$ and an arc from $y$ to $x$.

By forgetting the orientations of the arcs in $D$, we obtain an undirected graph called the underlying graph of $D$.

**Definition 2.3.** The underlying graph of $D$, denoted $\overline{D}$, is the undirected graph whose vertex set is $X$ and in which two vertices $x, y \in X$ are adjacent if and only if either $(x, y) \in A$ or $(y, x) \in A$.

Many definitions that are sensible for undirected graphs may now be imported to the digraph context.

**Definition 2.4.**

1. If $S \subseteq X$ and $|S| \geq 2$, then $S$ is biconnected if $\overline{D} \upharpoonright S$ is connected and $\overline{D} \upharpoonright (S \setminus \{s\})$ is connected for all $s \in S$.

2. A block in $D$ is a finite maximal biconnected set.

3. A connected component $C$ of $D$ is a Gallai tree if all blocks in $C$ are bicyclic, symmetric odd cycles, or complete symmetric graphs.

Throughout, we consider definable versions of the combinatorial notions for digraphs. For the fundamentals of descriptive set theory, we refer the reader to [Kec95].

**Definition 2.5.** Let $X$ be a standard Borel space. A Borel digraph on $X$ is a digraph $D = (X, A)$, where $A$ is Borel in the product topology on $X^2$.

If $Y$ is a Polish space, then a dicoloring $c$ of $D$ with colors from $Y$ is a Borel dicoloring if $c$ is a Borel function. The Borel dichromatic number of $D$, denoted $\chi_B(D)$, is the least $n \in \mathbb{N}$ such that there is a Borel dicoloring of $D$ with $n$ colors.

3. Basic Definable Digraph Combinatorics

In this section, we collect some basic results on the definable combinatorics of digraphs. The proof of the following proposition is similar to the proof of Proposition 4.3 in [KST99].
Proposition 3.1. Let $D$ be a Borel digraph on a standard Borel space $X$. Suppose that, for each $x \in X$, $d_{\min}(x) < \infty$. Then $D$ has a countable Borel dicoloring $c$ such that, for each $x \in X$, $c(x) \neq c(y)$ for any $y \in N_{\min}(x)$.

Now we show that there is a definable digraph analogue of the standard greedy algorithm upper bound on chromatic number.

Proof. Let $(U_n)_{n \in \mathbb{N}}$ be a basis for the topology on $X$. Define a Borel dicoloring $c : X \to \mathbb{N}$ by $c(x) = n_0$, where $n_0$ is the least $n \in \mathbb{N}$ such that $x \in U_n$ but $y \notin U_n$ for any $y \in N_{\min}(x)$. Then $c$ is as desired.

Proposition 3.2. Let $d \in \mathbb{N}$. Then for any Borel digraph $D$ on a standard Borel space $X$, if $d_{\min}(x) \leq d$ for all $x \in X$, then $\overline{\chi}_B(D) \leq d + 1$.

Proof. We proceed by induction on $d$. Suppose $d = 0$; then if $D$ is a Borel digraph on a standard Borel space $X$ such that $d_{\min}(x) = 0$ for all $x \in X$, it follows that $D$ has no dicycles, and so there is a Borel 1-dicoloring of $D$.

Now let $d \geq 0$, and assume that, for any Borel digraph $D$ on a standard Borel space $X$, if $d_{\min}(x) \leq d$ for all $x \in X$, then $\overline{\chi}_B(D) \leq d + 1$. Let now $D$ be a Borel digraph on a standard Borel space $X$ such that $d_{\min}(x) \leq d + 1$ for all $x \in X$. Apply the previous proposition to obtain a countable Borel dicoloring $c$ of $D$ such that, for all $x \in X$, $c(x) \neq c(y)$ for all $y \in N_{\min}(x)$. Then define $A_0 = \{x \in X : c(x) = 0\}$, and, having defined $A_n$, let $A_{n+1} = \{x \in X : c(x) = n + 1 \text{ and } N_{\min}(x) \cap (A_0 \cup \cdots \cup A_n) = \emptyset\}$. Put $A = \bigcup_{n \in \mathbb{N}} A_n$. Then $A$ is Borel.

We claim $A$ contains no dicycles. Otherwise, let $C = (x_0, x_1, \ldots, x_n = x_0)$ be a dicycle in $A$. If $n = 1$, then assume without loss of generality that $c(x_0) > c(x_1)$. Then $N_{\min}(x_0) \cap A_{c(x_1)} = \emptyset$, so that $x_0 \notin A_{c(x_1)}$, a contradiction. If $n > 1$, then let $\alpha = \max\{c(x_0), c(x_1), \ldots, c(x_{n-1})\}$. Then for some $i < n$, $c(x_i) = \alpha$. If $N_{\min}(x_i) = N^+(x_i)$, then since $c(x_i) > c(x_{i+1})$ and since $x_{i+1} \in A_{c(x_{i+1})}$, $x_i \notin A_{c(x_i)}$, a contradiction.

Similarly, if $N_{\min}(x_i) = N^-(x_i)$, then since $c(x_i) > c(x_{i-1})$ and since $x_{i-1} \in A_{c(x_{i-1})}$, $x_i \notin A_{c(x_i)}$.

We claim also that, for each $x \in X$, either $x \in A$ or there is $y \in N_{\min}(x) \cap A$. If $x \notin A$, then $x \notin A_{c(x)}$, so $c(x) > 0$ and $N_{\min}(x) \cap (A_0 \cup \cdots \cup A_{c(x)-1}) = \emptyset$. Therefore, if $x \in X \setminus A$, then $d_{B_{\min}}(x \setminus A) \leq d$. So by the inductive hypothesis, there is a Borel dicoloring $c' : (X \setminus A) \to \{0, \ldots, d\}$ of $D \mid (X \setminus A)$. Now define a Borel dicoloring $c_0$ of $D$ by $c_0(x) = c'(x)$ if $x \in X \setminus A$ and $c_0(x) = d+1$ if $x \in A$.

Next, we prove a definable version of degree-plus-one list coloring for digraphs.

Theorem 3.3. Let $D$ be a Borel digraph on a standard Borel space $X$, and assume $d_{\max}(x) < \infty$ for all $x \in X$. Let $Y$ be Polish, and let $L : X \to [Y]^{<\infty}$ be a Borel map such that $|L(x)| > d_{\max}(x)$ for all $x \in X$. Then $D$ has a Borel $L$-dicoloring, that is a Borel dicoloring $c'$ in which $c'(x) \in L(x)$ for each $x \in X$.

Proof. Since $\hat{D}$ is locally finite, there is a countable Borel proper coloring $c$ of $\hat{D}$. For each $n \in \mathbb{N}$, define $A_n = \{x \in X : c(x) = n\}$. Now let $<_Y$ be a Borel linear ordering on $Y$, and define $c_0 : A_0 \to Y$ by letting $c_0(x)$ be the $<_Y$-least element of $L(x)$ for each $x \in A_0$.

Now let $n \in \mathbb{N}$. For each $i \in \mathbb{N}$, let $B_i = \bigcup_{j \leq i} A_j$. Assume that, for each $i \leq n$, there is a list assignment $L^i : A_i \to [Y]^{<\infty}$ and a Borel function $c_i : B_i \to Y$ such that the following conditions obtain:

1. $c_i(x) \notin L^i(x)$ for all $x \in B_i$. 2. If $x \in B_i$, then $c_i(x)$ is $<_Y$-least in $L^i(x)$.

Then $c_0 \cup \cdots \cup c_n$ is a desired Borel $L$-dicoloring of $D$.
\( L^0(x) = L(x) \) for all \( x \in A_0 \);
\( c_i \mid B_j = c_i \) for all \( j < i \);
\( c_i \) is a Borel \( L \)-dicoloring of \( D \upharpoonright B_i \);
\( L^{i+1}(x) = L(x) \setminus \{ \alpha \in Y : \text{there exist } y, z \in B_i \text{ such that } y \in N^+(x), z \in N^-(x), \text{ and } c_i(y) = c_i(z) = \alpha \}; \) and
\( |L^{i+1}(x)| > d^{\max}_{D \upharpoonright (X \setminus B_i)}(x) \) for all \( x \in A_{i+1} \).

We proceed to define a list assignment \( L^{n+1} : A_{n+1} \to [Y]^{<\infty} \) and a Borel function \( c_{n+1} : B_{n+1} \to Y \) such that:

1. \( c_{n+1} \mid B_i = c_i \) for all \( i \leq n \);
2. \( c_{n+1} \) is a Borel \( L \)-dicoloring of \( D \upharpoonright B_{n+1} \);
3. \( L^{n+1}(x) = L(x) \setminus \{ \alpha \in Y : \text{there exist } y, z \in B_n \text{ such that } y \in N^+(x), z \in N^-(x), \text{ and } c_n(y) = c_n(z) = \alpha \}; \) and
4. \( |L^{n+1}(x)| > d^{\max}_{D \upharpoonright (X \setminus B_n)}(x) \) for all \( x \in A_{n+1} \).

Let \( x \in A_{n+1} \). Note that

\[
d^{\max}_{D \upharpoonright (X \setminus B_n)}(x) = \max\{d^+(x) - |N^+(x) \cap B_n|, d^-(x) - |N^-(x) \cap B_n|\}
\]

and that

\[
|L^{n+1}(x)| \geq |L(x)| - \min\{|N^+(x) \cap B_n|, |N^-(x) \cap B_n|\}.
\]

Without loss of generality, assume \( \max\{d^+(x) - |N^+(x) \cap B_n|, d^-(x) - |N^-(x) \cap B_n|\} = d^+(x) - |N^+(x) \cap B_n| \). Then

\[
|L(x)| - 1 - |N^+(x) \cap B_n| \leq |L(x)| - 1 - \min\{|N^+(x) \cap B_n|, |N^-(x) \cap B_n|\}
\]

So, \( d^{\max}_{D \upharpoonright (X \setminus B_n)}(x) < |L^{n+1}(x)| \). In particular, \( |L^{n+1}(x)| \geq 1 \). Then for any \( x \in B_{n+1} \), we may define \( c_{n+1}(x) = c_n(x) \) if \( x \notin A_{n+1} \) and \( c_{n+1}(x) \) is the \( \leq Y \)-least element of \( L^{n+1} \) if \( x \in A_{n+1} \). It is then easy to check that conditions (1), (3), and (4) above are satisfied by \( L^{n+1} \) and \( c_{n+1} \). To see that condition (2) is satisfied, assume for contradiction that there is a \( c_{n+1} \)-monochromatic dicycle \( C = (x_0, x_1, \ldots, x_k = x_0) \) in \( B_{n+1} \). If \( k = 1 \), then without loss of generality, assume \( c(x_0) > c(x_1) \) since \( c \) is a proper coloring of \( D \). Then \( x_1 \in N^+(x_0) \cap N^-(x_0) \cap A_{c(x_0)}, \) so that \( L^c(x_0)(x_0) \) does not contain \( c(x_1) \), a contradiction since \( c_{n+1}(x_0) = c_{n+1}(x_1) \). If \( k > 1 \), then there is \( i < k \) such that \( c(x_i) > c(x_{i-1}), c(x_{i+1}) \). Then \( \alpha := c_{n+1}(x_{i-1}) = c_{n+1}(x_{i+1}) \) is not an element of \( L^c(x_i) \), a contradiction since \( c_{n+1}(x_i) = \alpha \).

Now define \( c' : X \to Y \) by \( c' = \bigcup_{n \in \mathbb{N}} c_n \). Then \( c' \) is a Borel \( L \)-dicoloring of \( D \).

### 4. Measurable Brooks’s Theorem for Dicolorings

In this section, we prove Theorem 1.6. The proof relies heavily upon the one-ended spanning forest technique developed by Conley, Marks, and Tucker-Drob in [CMT16].

**Definition 4.1.** Let \( f \) be a function on a set \( X \). Then \( f \) is **one-ended** if there is no infinite sequence \( (x_n)_{n \in \mathbb{N}} \) such that, for each \( n \in \mathbb{N} \), \( f(x_{n+1}) = x_n \).
Note that one-ended functions do not have fixed points.

We now show that, if the underlying graph of a digraph $D$ admits a Borel one-ended function, then $D$ is Borel degree-list dicolorable. The proof resembles that of Lemma 3.9 in [CMT16].

**Theorem 4.2.** Let $D$ be a Borel digraph on a standard Borel space $X$, let $B \subseteq X$ be Borel, and let $f : B \to X$ be a one-ended Borel function whose graph is contained in $\bar{D}$. Let $d \in \mathbb{N}$ be such that $d^\max(x) \leq d$ for all $x \in X$. Let $Y$ be Polish, and let $L : X \to [Y]^{<\infty}$ be a Borel function such that $|L(x)| \geq d^\max(x)$ for all $x \in B$. Then $D \upharpoonright B$ has a Borel $L$-dicoloring.

**Proof.** First, note that we may assume without loss of generality that there are no isolated vertices in $D$ (if there are isolated vertices, then select a color $y \in Y$, and assign the color $y$ to each isolated vertex).

Now, for each $n \in \mathbb{N}$, write $f^n[B] = \{x \in X : \text{there exist } x_1, x_2, \ldots, x_n \in B \text{ such that } f(x_1) = x \text{ and } f(x_{i+1}) = f(x_i) \text{ for all } i < n\}$. Let $B_n = B \cap (f^n[B] \setminus f^{n+1}[B])$. Note that, if $n \neq m$, then $B_n \cap B_m = \emptyset$. We claim that $B = \bigcup_{n \in \mathbb{N}} B_n$. Assume for contradiction that there is $x \in B \setminus \bigcup_{n \in \mathbb{N}} B_n$. Then $x \in f^n[B]$ for all $n \in \mathbb{N}$.

So the set $f^{-N}(x) = \{y \in B : \text{for some } n \geq 1, \text{ there are } x_1, x_2, \ldots, x_{n-1} \in B \text{ such that } f(x_1) = x, f(x_{i+1}) = x_i \text{ for all } i < n, \text{ and } f(y) = x_{n-1}\}$ is an infinite, finitely branching tree with root $x$. By König’s lemma, there is an infinite branch $(y_n)_{n \in \mathbb{N}}$ through $f^{-N}(x)$. Then for each $n \in \mathbb{N}$, $f(y_{n+1}) = y_n$, contradicting that $f$ is one-ended.

We proceed to $L$-dicolor $B$ one layer at a time. First, we color $B_0$. Note that $\bar{D} \upharpoonright B_0$ is an undirected graph in which each vertex has degree at most $2d$, since $d^\max(x) \leq d$ for all $x \in B$. So, by Proposition 4.2 in [KST99], there is a Borel maximal $\bar{D}$-independent set $B_0^0 \subseteq B_0$. Note, let $0 < j < 2d$, and suppose $B_0^j \subseteq B_0$ has been defined for each $j < i$ such that $B_0^j$ is a Borel maximal $\bar{D}$-independent set in $B_0 \setminus (B_0^0 \cup B_0^1 \cup \cdots \cup B_0^{j-1})$. Then let $B_0^j+1$ be a Borel maximal $\bar{D}$-independent set in $B_0 \setminus (B_0^0 \cup B_0^1 \cup \cdots \cup B_0^j)$. Finally, note that, if $x \in B_0 \setminus (B_0^0 \cup B_0^1 \cup \cdots \cup B_0^{2d})$, then $d^\max_{\bar{D}(B_0 \setminus (B_0^0 \cup B_0^1 \cup \cdots \cup B_0^{2d})\cup \{x\})} = d^\max_{\bar{D}(B_0 \setminus (B_0^0 \cup B_0^1 \cup \cdots \cup B_0^{2d}))} = d^\max_{\bar{D}(B_0 \setminus (B_0^0 \cup B_0^1 \cup \cdots \cup B_0^{2d}))}$. Then $B_0^j+1 := B_0 \setminus (B_0^0 \cup B_0^1 \cup \cdots \cup B_0^{j})$ is a Borel $\bar{D}$-independent set. For each $i \leq 2d + 1$, write $B_0^{<i} = \emptyset$ if $i = 0$ and $B_0^{<i} = \bigcup_{j < i} B_0^j$ if $i > 0$, and write $B_0^i = \bigcup_{j \leq i} B_0^j$.

Now note that, for each $x \in B_0^i$, $d^\max_{\bar{D}(B_0^i)}(x) = 0$. Since $D$ has no isolated vertices, and since $|L(x)| \geq d^\max(x)$ for each $x \in X$, it follows that $|L(x)| \geq 1$ for each $x \in B_0^i$. So, by Theorem 3.3, there is a Borel $L$-dicoloring $c_0^i$ of $D \upharpoonright B_0^i$. Now let $1 \leq i \leq 2d$, and suppose Borel $L$-dicolorings $c_0^j : B_0^{\leq j} \to Y$ have been defined for all $j \leq i$ so that the following conditions hold:

- If $j' < j$, then $c_0^j | (B_0^{\leq j'}) = c_0^{j'}$; and
- For any $1 \leq j' < i$ and for any $x \in B_0^{j'}$, if $x$ has an out-neighbor $y \in B_0^{<j'}$ and an in-neighbor $z \in B_0^{<j}$ with $\alpha := c_0^{j'-1}(y) = c_0^{j'-1}(z)$, then $c_0^{j'}(x) \neq \alpha$.

Then for each $x \in B_0^{i+1}$, let

$L_0^{i+1}(x) = L(x) \setminus \{\alpha \in Y : \text{there are } y, z \in B_0^{\leq j} \text{ such that } c_0^i(y) = c_0^i(z) = \alpha\}.$
Then, since $x$ has at least one neighbor, namely $f(x)$, not contained in $B_0$,
\[
|L_0^{i+1}(x)| \geq |L(x)| - (d^{\text{max}}(x) - 1) \\
\geq d^{\text{max}}(x) - (d^{\text{max}}(x) - 1) \\
= 1,
\]
so that $d^{\text{max}}_{D|B_0^{i+1}}(x) < |L_0^{i+1}(x)|$. Therefore, by Theorem 3.3, there is a Borel $L_0^{i+1}$-dicoloring $c$ of $D \upharpoonright B_0^{i+1}$. Define $c^{i+1} : B_0^{i+1} \to Y$ by $c^{i+1}(x) = c(x)$ if $x \in B_0^{i+1}$ and $c^{i+1}_0(x) = c^i_0(x)$ otherwise. Finally, define $c_0 := c^{2d+1}_0$.

We now proceed to color $B_1$. The procedure is similar to that for coloring $B_0$; we construct $\bar{D}$-independent sets $B_0^1, B_1^1, \ldots, B_1^{2d+1}$ as above and color one set at a time to obtain a coloring $c_0$ on $D \upharpoonright B_1$. However, in this case, if $x \in B_1^0$, then $L_0^1(x) := L(x) \{ \alpha \in Y : \text{there exist } y, z \in B_0 \text{ such that } y \in N^+(x), z \in N^-(x), \text{ and } c_0(y) = c_0(z) = \alpha \}$. Then again $d^{\text{max}}_{D|B_1^0}(x) < |L_0^1(x)|$, so that $D \upharpoonright B_1^0$ may be $L_0^1$-dicolored.

Now let $c = \bigcup_{n \in \mathbb{N}} c_n$. Then $c$ is Borel. Assume for contradiction that there is a $c$-monochromatic cycle $C = (x_0, x_1, \ldots, x_k = x_0)$ in $D$. If $k > 1$, then let $n \in \mathbb{N}$ be maximal with $C \cap B_n \neq \emptyset$, and let $j < 2d + 1$ be maximal with $C \cap B_j \neq \emptyset$. Then there is $i \leq k$ such that $x_i \in B_j$. It follows that $x_{i-1}, x_{i+1} \notin B_n$. Then either $x_{i-1} \in B_m$ for some $m < n$, or $x_{i+1} \in B_j$ for some $j' < j$, and similarly for $x_{i+1}$. In any case, it follows that $\alpha := c(x_{i-1}) = c(x_{i+1})$ is not an element of $L_0^1(x_i)$, a contradiction since $c(x_i) = \alpha$. If $k = 1$, then the argument is similar. Thus, $c$ is a Borel $L$-dicoloring of $D$.

We will use the following proposition of Conley, Marks, and Tucker-Drob several times to construct one-ended Borel functions.

**Proposition 4.3** ([CMT16], Proposition 3.1). *Let $G$ be a locally finite Borel graph on a standard Borel space $X$, and let $A \subseteq X$ be Borel. Then there is a one-ended Borel function $f : ([A|_G \setminus A) \to [A|_G$ whose graph is contained in $G$.***

In our next proposition, we show that digraphs of bounded maximum degree $d$ in which some vertices have small degree are Borel $d$-dicolorable. In the construction of the dicoloring, we first reserve an independent set of small-degree vertices and color the vertices outside of this set using a one-ended function. Then since each reserved vertex has small degree, there is at least one color which does not appear among both the out-neighbors and the in-neighbors of the vertex, so that the initial dicoloring can be extended to the reserved vertices. The proof resembles the first part of the proof of Theorem 1.2 in [CMT16].

**Proposition 4.4.** *Let $D$ be a Borel digraph on a standard Borel space $X$, let $d \in \mathbb{N}$ be such that $d^{\text{max}}(x) \leq d$ for all $x \in X$, and let $B = \{ x \in X : \text{deg}_{\bar{D}}(x) < 2d \}$. Then $D \upharpoonright [B]_{\bar{D}}$ has Borel dichromatic number at most $d$.*

**Proof.** Since each vertex in $D$ has maximum degree at most $d$, each vertex in $\bar{D}$ has degree at most $2d$. Therefore, by Proposition 4.2 in [KST99], there is a Borel maximal $\bar{D}$-independent set $B' \subseteq B$. In particular, $[B']_{\bar{D}} = [B]_{\bar{D}}$, and so by Proposition 4.3, there is a one-ended Borel function $f : ([B]_{\bar{D}} \setminus B') \to [B]_{\bar{D}}$ whose graph is contained in $\bar{D}$. Then by Theorem 4.2, $D \upharpoonright ([B]_{\bar{D}} \setminus B')$ has a Borel $d$-dicoloring $c$. 
Now we extend $c$ to a function $c': B \to \{1, 2, \ldots, d\}$ by letting $c'(x) = c(x)$ if $x \in B \setminus B'$; if $x \in B'$, then $c'(x)$ is defined to be the least color $i$ such that there are no $y \in N^+(x) \cap (B \setminus B')$ and $z \in N^-(x) \cap (B \setminus B')$ with $c(y) = c(z) = i$. Such an $i$ exists since $\deg_D(x) < 2d$ implies $d_{\text{min}}(x) < d$, so that at most $d - 1$ colors appear in $c[N^+(x)] \cap c[N^-(x)]$. It is then easy to check that $c'$ is a Borel $d$-dicoloring of $D \upharpoonright [B]_D$. \hfill $\square$

Next, we move toward showing that it is possible to $d$-dicolor connected components of digraphs of bounded maximum degree $d$ which are not Gallai trees. We first need the following definition.

**Definition 4.5.** Let $D$ be a digraph on a set $X$. A block $S$ of $D$ is **good** if $D \upharpoonright S$ is not a directed cycle, a symmetric odd cycle, or a complete symmetric graph; otherwise, $S$ is **bad**.

The next proposition of Harutyunyan and Mohar shows the following: Suppose $D$ is a digraph of bounded maximum degree $d$ such that all vertices except one vertex $x$ have been dicolored by a function $c$. If $x$ has small degree, or if $x$ has either two out-neighbors or two in-neighbors that receive the same color from $c$, then $c$ can be easily extended to $x$. Otherwise, the proposition shows that, by uncoloring a neighbor $y$ of $x$ and then coloring $x$ with the color $y$ previously had, we obtain a new dicoloring $c'$ of $D \upharpoonright (X \setminus \{y\})$.

**Proposition 4.6** ([HM11], Lemma 2.2). Let $D$ be a digraph on a set $X$ such that, for some $d \in \mathbb{N}$, $d^{\text{max}}(x) = d^{\text{min}}(x) = d$ for all $x \in X$. Let $x \in X$, and suppose $c$ is a $d$-dicoloring of $D \upharpoonright (X \setminus \{x\})$ such that, for each $1 \leq i \leq d$, $x$ has both an out-neighbor and an in-neighbor of color $i$. Let $C = (x_0, x_1, \ldots, x_n = x_0)$ be a cycle in $\tilde{D}$ with $x = x_0$. Let $y \in X$ be adjacent to $x$ in $\tilde{D}$, and define $c' : (X \setminus \{y\}) \to \{1, 2, \ldots, d\}$ by $c'(x) = c(y)$ and $c'(z) = c(z)$ if $z \neq x$. Then $c'$ is a dicoloring of $D \upharpoonright (X \setminus \{x_1\})$.

Before stating our next theorem, we need the following definition.

**Definition 4.7.** Let $D$ be a digraph on a set $X$, and let $S \subseteq X$ be finite. Then the **boundary of** $S$, denoted $\partial S$, is the set $\partial S = \{x \in X : x \not\in S \text{ but there exists } y \in S \text{ such that } x \neq y\}$.

Now we show that the connected components of a digraph that are not Gallai trees are possible to dicolor. The proof of the following theorem is similar to the proof of Theorem 4.1 in [CMT16]. The argument involves reserving a set $B''$ of points that belong to good blocks which are well-separated. Once the points outside $B''$ have been dicolored using a one-ended function, it is possible to color the points inside $B''$ by repeated applications of Proposition 4.6.

**Theorem 4.8.** Let $D$ be a Borel digraph on a standard Borel space $X$, and let $d \in \mathbb{N}$ be such that $d^{\text{max}}(x) = d^{\text{min}}(x) = d$ for all $x \in X$. Let $B = \{x \in X : x \in \partial D \text{ is a Gallai tree}\}$. Then $D \upharpoonright B$ has a Borel $d$-dicoloring.

**Proof.** Let $[E_{\tilde{D}}]^{\prec\infty} := \{S \in [X]^{\prec\infty} : S \text{ is contained in a connected component of } \tilde{D}\}$, and let

$$A = \{S \in [E_{\tilde{D}}]^{\prec\infty} : S \text{ is a good block}\}.$$

Define $G_1$ to be the intersection graph on $[E_{\tilde{D}}]^{\prec\infty}$, so that $S, T \in [E_{\tilde{D}}]^{\prec\infty}$ are $G_1$-adjacent if and only if $S \cap T \neq \emptyset$. Then by Proposition 2 of [CMT16], there is a
countable Borel proper coloring $c_I$ of $G_I$. Let now

$$A' = \{ S \in A : c_I(S \cup \partial S) \leq c_I(T \cup \partial T) \text{ for all } T \in A \text{ in the same } \tilde{D} \text{-component as } S \},$$

and define $B' = \bigcup A'$.

We now prove several claims about $B'$. First, we show that, if $x \in B'$, then there is a unique $S \in A'$ such that $x \in S$. Indeed, let $x \in B'$; then there is some $S \in A'$ such that $x \in S$. If $S' \in A'$ is distinct from $S$, then $x \in S'$ implies $S \cap S' \neq \emptyset$. Then $c_I(S \cup \partial S) \neq c_I(S' \cup \partial S')$; however, since $S, S'$ are in the same connected component of $\tilde{D}, c_I(S \cup \partial S) = c_I(S' \cup \partial S')$, a contradiction.

Now we claim that $[x]_{\tilde{D}\{B'} = [x]_{\tilde{D}\{S}$, where $S \in A'$ is unique with $x \in S$. Then clearly $[x]_{\tilde{D}\{B'} \supseteq [x]_{\tilde{D}\{S}$. To see $[x]_{\tilde{D}\{B'} \subseteq [x]_{\tilde{D}\{S}$, we show that there is no $\tilde{D}$-path through $B'$ starting at $x$ that contains a point not in $S$. Indeed, assume for contradiction that $(x = x_0, x_1, \ldots, x_n = y)$ is a path through $B'$ such that, for some $0 < i \leq n$, $x_i \notin S$. Let $i > 0$ be least such that $x_i \notin S$. Then $x_{i-1} \in S$, so that $x_i \in \partial S$. Let $T \in A'$ be such that $x_i \in T$. Then $(S \cup \partial S) \cap (T \cup \partial T) \neq \emptyset$, so that $c_I(S \cup \partial S) = c_I(T \cup \partial T)$, a contradiction since $S, T$ are in the same $\tilde{D}$-component.

Now it is clear that each connected component of $\tilde{D} \upharpoonright B'$ is a (finite) good block. Hence, $B' \subseteq B$. Also, since each connected component of $B$ contains some element of $A$ and hence some element of $A'$, $B'$ meets each connected component of $\tilde{D} \upharpoonright B$. Now let $< \chi$ be a Borel linear ordering of $X$. It follows from the proof of Theorem 2.1 in [HM11] that, if $D \upharpoonright S$ is a good block, then there is some undirected cycle $C \subseteq S$ such that $D \upharpoonright C$ is a good block. So, let $B'' = \{ x \in B' : x = \chi$-least in the unique $S \in A'$ such that $x \in S$ and there is an undirected cycle $C \subseteq S$ with $x \in C$ and $D \upharpoonright C$ a good block}. Then $[B'']_{\tilde{D}} = [B']_{\tilde{D}} = B$.

So, by Proposition 4.3, there is a one-ended Borel function $f : (B \setminus B'') \to B$. Then by Theorem 4.2, there is a Borel $d$-dicoloring $c$ of $D \upharpoonright (B \setminus B'')$.

We proceed to define a Borel $d$-dicoloring $c'$ of $D \upharpoonright B$. For each $x \in B''$, let $S_x \in A'$ be the unique $S \in A'$ such that $x \in S$. Note that, if $x, y \in B''$ are distinct, then $(S_x \cup \partial S_x) \cap (S_y \cup \partial S_y) = \emptyset$. Now fix $x \in B''$. Let $C_x = (x_0, x_1, \ldots, x_n = x_0)$ be an undirected cycle in $S_x$ with $x = x_0$ such that $D \upharpoonright C$ is a good block. Note that, if $x$ has either two out-neighbors or two in-neighbors that receive the same color from $c$, then $c$ can be extended to $x$ by letting $c(x)$ be the least color not appearing on both sides of $x$. So, assume all colors appear on both sides of $x$. We will use Proposition 4.6 to shift the colors within each $C_x$. We first need the following claim, which guarantees that shifting the coloring around different points of $B''$ does not produce monochromatic dicycles:

**Claim.** Let $x, y \in B''$ be distinct with all colors appearing on both sides of $x, y$. Let $x'$ be a neighbor of $x$, and let $y'$ be a neighbor of $y$. Then the coloring $c'$ given by $c'(z) = c(z)$ if $z \neq x, y, x', y'$, $c'(x) = c(x')$, and $c'(y) = c(y')$, with $x', y'$ becoming uncolored, is a dicoloring.

**Proof of claim.** If $C$ is a $c'$-monochromatic dicycle, then $C$ cannot contain $x'$ or $y'$, since these vertices are uncolored. Further, $C$ must contain either $x$ or $y$; otherwise, there is a $c$-monochromatic dicycle, contradicting that $c$ is a dicoloring. Suppose $x \in C$. Then $c'(x) = c(x')$; since all colors appear in $c[N^+(x)]$ and $c[N^-(x)]$, it follows that $c'(x) \neq c'(z)$ for all but one neighbor $z$ of $x$. Further, since $(S_y \cup \partial S_y) \cap (S_x \cup \partial S_x) = \emptyset$, $y, y'$ are not neighbors of $x$, and so the only neighbor of $x$ whose color changes in passing from $c$ to $c'$ is $x'$. Therefore, any dicycle passing through $x$ either contains points of different colors or contains an
uncolored point. So, no monochromatic dicycle contains \( x \). A similar argument shows that no monochromatic dicycle contains \( y \). \( \square \)

So, by repeatedly applying Proposition 4.6, we can shift colors in \( C_x \) for each \( x \in B'' \) without producing dicycles. If any color shift produces an uncolored point \( x' \) for which either two out-neighbors or two in-neighbors of \( x' \) receive the same color, then the current coloring can immediately be extended to \( x' \).

By the proofs of Lemmas 2.4 and 2.5 in [HMT11], since \( D \upharpoonright C_x \) is a good block for each \( x \in B'' \), there is some shift of colors in \( C_x \) that yields such an \( x' \); define the first such \( x' \) that is obtained in this way to be \( g(x) \), and let \( c_x \) be the shift of \( c \) within \( C_x \) that results in \( g(x) \) becoming uncolored. By the claim above, \( \bigcup_{x \in B''} c_x \) is a \( d \)-dicoloring of \( D \upharpoonright (X \setminus \{g(x) : x \in B''\}) \). Also, for each \( x \in B'' \), \( c_x \) can be extended to a function \( c'_x \) defined on \( g(x) \) without producing monochromatic directed cycles. By the claim above, \( c' := \bigcup_{x \in B''} c'_x \) is a Borel \( d \)-dicoloring of \( D \).

We next state a result of [CMT16] that shows that, modulo a null set or a meager set, an undirected acyclic graph in which no connected component has 0 or 2 ends admits a one-ended Borel function. We first recall the following definition.

**Definition 4.9.** Let \( G \) be a locally finite graph on a set \( X \). A **ray** in \( G \) is an infinite sequence \( (x_n)_{n \in \mathbb{N}} \) of pairwise-adjacent vertices in \( G \) such that \( x_n \neq x_m \) whenever \( n \neq m \). Two rays \( r_0, r_1 \) in \( G \) are **end-equivalent** if, whenever \( S \subseteq X \) is finite, \( r_0 \) and \( r_1 \) eventually lie in the same connected component of \( G \upharpoonright (X \setminus S) \). End-equivalence is an equivalence relation on the set of rays; the equivalence classes are called **ends**.

**Theorem 4.10** ([CMT16], Theorem 1.5). Let \( G \) be a locally finite acyclic graph on a standard Borel space \( X \). Assume no connected component of \( G \) has either 0 or 2 ends.

1. For any Borel probability measure \( \mu \) on \( X \), there are a \( \mu \)-conull Borel set \( B \) and a one-ended Borel function \( f : B \to X \) whose graph is contained in \( G \).

2. For any Polish topology \( \tau \) compatible with the Borel structure on \( X \), there is a \( \tau \)-comeager Borel set \( B \) and a one-ended Borel function \( f : B \to X \) whose graph is contained in \( G \).

The proof of the following theorem resembles the proof of Theorem 4.2 in [CMT16].

**Theorem 4.11.** Let \( D \) be a Borel digraph on a standard Borel space \( X \). Suppose there is \( d \in \mathbb{N} \) such that \( d_{\text{max}}(x) = d_{\text{min}}(x) = d \) for all \( x \in X \). Assume that \( \hat{D} \) has no finite connected components that are Gallai trees and no infinite connected components that are 2-ended Gallai trees. Then:

1. For any Borel probability measure \( \mu \) on \( X \), there are a \( \mu \)-conull, \( \hat{D} \)-invariant Borel set \( B \) such that \( D \upharpoonright B \) is Borel \( d \)-dicolorable.

2. For any Polish topology \( \tau \) compatible with the Borel structure on \( X \), there is a \( \tau \)-comeager, \( \hat{D} \)-invariant Borel set \( B \) such that \( D \upharpoonright B \) is Borel \( d \)-dicolorable.

**Proof.** First, let \( A = \{ x \in X : [x]_{\hat{D}} \text{ is not a Gallai tree} \} \). Then by Theorem 4.8, \( D \upharpoonright A \) has a Borel \( d \)-dicoloring. So we may assume without loss of generality that each connected component of \( \hat{D} \) is an infinite Gallai tree that is not 2-ended.

Let \( Y = \{ S \in [X]<\infty : S \text{ is a block in } D \} \), and consider the graph \( G \) on \( Z := X \sqcup Y \) such that there is an edge between \( x \in X \) and \( S \in Y \) if and only if \( x \in S \). Note
that, if \( S, S' \in \mathcal{Y} \) are distinct, then \( |S \cap S'| \leq 1 \); otherwise, \( S \cup S' \) is biconnected, contradicting that \( S, S' \) are maximal biconnected sets. Also, \( G \) is an acyclic graph; it is clear that \( G \) does not contain any odd cycles. Assume for contradiction that \( G \) contains an even cycle \((x_0, S_0, x_1, S_1, \ldots, x_n, S_n, x_{n+1} = x_0)\). Then \( \bigcup_{0 \leq i \leq n} S_i \) is a biconnected set, again contradicting the maximality of \( S_0, S_1, \ldots, S_n \).

Now, if \( x \in X \), then \([x]_D\) is an infinite Gallai tree, so that each block in \([x]_D\) is bad. Let \( S \) be any maximal biconnected set containing \( x \); then \( S \) is a block, and so \( x \) has at least one neighbor in \( G \). Further, \( G \) is locally finite, since \( \hat{D} \) is locally finite and each block containing \( x \) is uniquely determined by \( x \) and one of its finitely many neighbors. In addition, each connected component of \( G \) is infinite, since each connected component of \( \hat{D} \) is infinite and any path in \( \hat{D} \) corresponds to a path in \( G \); also, no connected component of \( G \) is 2-ended, since no connected component of \( \hat{D} \) is 2-ended.

So, by Theorem 4.10, there is, on either a \( \mu \)-conull set or a \( \tau \)-comeager set \( B \), a one-ended Borel function \( f \) whose graph is contained in \( G \). Now let \( <_X \) be a Borel linear ordering of \( X \), and let \( <_Y \) be a Borel linear ordering of \( Y \). Then define a Borel function \( g : X \to Y \) by letting \( g(x) \) be the \( <_Y \)-least block in \( Y \) containing \( x \).

Let also \( g' : X \to Y \) be a Borel function defined by \( g'(x) = g(x) \) if \( x \notin f(g(x)) \) and \( g'(x) = f(g(x)) \) if \( x \in f(g(x)) \). Then define a one-ended Borel function \( \hat{f} : B \to X \) by letting \( \hat{f}(x) \) be the (unique) neighbor of \( x \) on the \( <_X \)-least path from \( x \) to a point in \( f(g'(x)) \). So by Theorem 4.2, \( D \upharpoonright B \) has a Borel \( d \)-dicoloring.

Finally, we prove Theorem 1.6. We restate the theorem for convenience.

**Theorem 4.12.** Let \( D \) be a Borel digraph on a standard Borel space \( X \). Suppose there is \( d \geq 3 \) such that \( d^{\max}(x) \leq d \) for all \( x \in X \), and assume \( D \) does not contain the complete symmetric graph on \( d \) vertices. Then:

1. For any Borel probability measure \( \mu \) on \( X \), there is a \( \mu \)-measurable dicoloring of \( D \) with \( d \) colors.
2. For any Polish topology \( \tau \) compatible with the Borel structure on \( X \), there is a \( \tau \)-Baire-measurable dicoloring of \( D \) with \( d \) colors.

**Proof.** By Proposition 4.4, we may assume that \( d^{\max}(x) = d^{\min}(x) = d \) for all \( x \in X \). Note that, if all the vertices in a finite Gallai tree have out-degree and in-degree equal to \( d \), then the Gallai tree is the complete symmetric graph on \( d \) vertices. Also, if all the vertices in an infinite 2-ended Gallai tree have out-degree and in-degree equal to \( d \), then either \( d = 1 \) and the tree is a one-directional bi-infinite line, or \( d = 2 \) and the tree is a symmetric bi-infinite line. Since \( d \geq 3 \), it follows from Theorem 4.11 that \( D \) has a measurable \( d \)-dicoloring.

**Acknowledgments**

We would like to thank Andrew Marks for many helpful conversations about the content of this note.

**References**


<table>
<thead>
<tr>
<th>Reference</th>
<th>Title</th>
</tr>
</thead>
</table>