BREDON HOMOLOGY OF REPRESENTATION SPHERES, $G = C_{2^n}$

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1. Introduction

Computing the ($\mathbb{Z}$-graded) slice E_2 pages for (localizations of quotients of) $BP^{((C_{2^n}))}$ amounts to computing the Bredon homology of (actual, as opposed to virtual) C_{2^n}-representation spheres. In the actual case, each such representation sphere has a particularly simply G-CW structure that makes this computation straightforward. We present this computation at $n = 2$, i.e. the case $G = C_4$. We fix a generator γ of C_4. For $G = C_4$, the real representation ring $RO(C_4)$ is, additively, $\mathbb{Z}\{1, \sigma, \lambda\}$, where σ is the sign representation of C_4, and λ is the 2-dimensional representation where γ acts on the complex plane via multiplication by $e^{2\pi i/4}$. In particular, as an abelian group, $RO(C_4) = \mathbb{Z}\{1, \sigma, \lambda\}$, and in this case $RO(C_4)$ and $JO(C_4)$ coincide.

In particular, each actual representation of C_4 is of the form $a + b\sigma + c\lambda$ for $a, b, c \geq 0$. In section 2, we describe an equivariant cell structure for $S^{a+b\sigma+c\lambda}$, in section 3 we record some lemmas about cyclic G-modules, in section 4 we carry out the computation in the orientable case, and in section ? we carry out the computation in the non-orientable case.

2. Cell structures for S^V

By definition, the Bredon homology of representation spheres with a trivial action is concentrated in degree 0, i.e.

$$\underline{H}_k(S^n; \underline{\mathbb{Z}}) = \begin{cases} 0 & n \neq k \\ \underline{\mathbb{Z}} & n = k \end{cases}$$

Moreover, by use of suspension isomorphisms

$$\underline{H}_k(S^{a+b\sigma+c\lambda}; \underline{\mathbb{Z}}) \cong \underline{H}_{k-a}(S^{b\sigma+c\lambda}; \underline{\mathbb{Z}})$$

so it suffices to consider $V = b\sigma + c\lambda$ for $b, c \geq 0$. We give the following G-CW structures as a 0-skeleton followed by a list of cofiber sequences.

- S^σ:
  - S^0
  - ${C_4/C_2}_+ \to S^0 \to S^\sigma$
- S^λ:
  - S^0
  - $C_{4+} \to S^0 \to (S^\lambda)^{(1)}$
  - $C_{4+} \wedge S^1 \to (S^\lambda)^{(1)} \to S^\lambda$

For S^σ, the map $C_4/C_{2+} \to S^0$ is adjoint to the C_2-map $\mathrm{id} : S^0 \to S^0$. For S^λ, the map $C_{4+} \to S^0$ is adjoint to the nonequivariant map $\mathrm{id} : S^0 \to S^0$, and the 1-skeleton $(S^\lambda)^{(1)}$ is also called the spoke sphere. The map $C_{4+} \wedge S^1 \to (S^\lambda)^{(1)}$ is adjoint to the nonequivariant map $S^1 \to (S^\lambda)^{(1)}$ that traces up one spoke and then down an adjacent spoke.

In the general case $S^{a\sigma+b\lambda}$, we simply smash these cofiber sequences together. As an example, we give the cell structure for $S^{2\sigma+2\lambda}$:

- $S^{2\sigma+2\lambda}$:
  - S^0
  - $C_4/C_{2+} \to S^0 \to S^\sigma$
  - $C_4/C_{2+} \wedge S^\sigma \to S^\sigma \to S^{2\sigma}$
  - $C_{4+} \wedge S^{2\sigma} \to S^{2\sigma} \to (S^\lambda)^{(1)} \wedge S^{2\sigma}$
  - $C_{4+} \wedge S^1 \wedge S^{2\sigma} \to (S^\lambda)^{(1)} \wedge S^{2\sigma} \to S^\lambda \wedge S^{2\sigma}$
  - $C_{4+} \wedge S^\lambda \wedge S^{2\sigma} \to S^\lambda \wedge S^{2\sigma} \to (S^\lambda)^{(1)} \wedge S^\lambda \wedge S^{2\sigma}$
  - $C_{4+} \wedge S^1 \wedge S^\lambda \wedge S^{2\sigma} \to (S^\lambda)^{(1)} \wedge S^\lambda \wedge S^{2\sigma} \to S^\lambda \wedge S^\lambda \wedge S^{2\sigma} = S^{2\sigma+2\lambda}$

An important point here is that these cofiber sequences actually do describe a G-CW structure (not just a *Rep*(G)-CW structure) because the lefthand term in each cofiber sequence supports a Frobenius isomorphism to an actual G-CW cell. For example, we have

$$C_{4+} \wedge S^\lambda \wedge S^{2\sigma} \simeq C_{4+} \wedge S^3$$

One should expect to have to keep track of these identifications when using these cell structures to compute Bredon homology (indeed, they can introduce signs in the boundary maps). However, as we will show, all the boundary maps are completely determined by the underlying homology and the fact that, in these cell structures, each dimension has a single G-CW cell.

3. An algebraic lemma

Let G be a finite abelian group and suppose we have a chain complex of G-modules

$$0 \leftarrow C_0 \leftarrow C_1 \leftarrow C_2 \leftarrow \cdots \leftarrow C_n \leftarrow 0$$

By applying the fixed point Mackey functor construction to this chain complex, we get a chain complex of Mackey functors, whose homology Mackey functors $\underline{H}_*$ we may call the Bredon homology of C_*. Suppose we know the following:

1. $\underline{H}_i(G/e) = 0$ for all $i < n$. (i.e. the underlying homology vanishes except in degree n).
2. For each i, $C_i \cong \mathbb{Z}[G/K_i]$ as a G-module, for some $K_i \subset G$.

In this situation, we have the following:

Lemma 3.1. *For $i < n$, $\ker(d_i)$ is a cyclic G-module on a generator that is fixed by K_{i+1}.*

Proof. For each $i < n$, the differential $d_{i+1} : C_{i+1} \cong \mathbb{Z}[G/K_{i+1}] \to \ker(d_i)$, which is adjoint to an K_{i+1}-equivariant map $\mathbb{Z} \to \ker(d_i)$, i.e. a choice of element $x_i \in (\ker(d_i))^{K_{i+1}}$. By (1), d_{i+1} is surjective, hence x_i is a G-module generator for $\ker(d_i)$. □

Hence, the differential d_{i+1} is determined by some G-module generator $x_i \in (\ker(d_i))^{K_{i+1}}$. We show that all such choices give the same homology:

Lemma 3.2. *In the above situation, the choice of G-module generator $x_i \in (\ker(d_i))^{K_{i+1}}$ does not change $\underline{H}_i$ or $\underline{H}_{i+1}$.*

Proof. Let $x_i, x_i' \in (\ker(d_i))^{K_{i+1}}$ be two G-module generators of $\ker(d_i)$. There is a G-module automorphism ϕ of $\ker(d_i)$ such that $\phi(x_i) = x_i'$. Indeed, since x_i is a G-module generator, $x_i' = \sum_j a_j g_j(x_i)$ for some $g_j \in G$ and $a_j \in \mathbb{Z}$, hence we let ϕ be the map $\sum_j a_j g_j$; this is a G-module endomorphism since G is abelian. It is surjective since x_i' is a G-module generator, and therefore an automorphism as $\ker(d_i)$ is free abelian of finite rank.

The two differentials determined by x_i and x_i' respectively thus differ by a G-equivariant automorphism of $\ker(d_i)$, and the lemma follows by functoriality. □

Remark 3.3. The chain complexes computing $\underline{H}_*(S^V)$ have the above properties because the underlying homology is concentrated in degree $|V|$, and the cell structures in Section 2 have a single cell in each dimension.

4. The orientable case

An actual C_4 representation $V = b\sigma + c\lambda$ is orientable if and only if b is even, hence we assume for the remainder of the section that $b = 2k$ for $k \geq 0$.

4.1. **The chain complex of G-modules.**

We use sections 2 and 3 to determine the chain complex of C_4-modules that determines $\underline{H}_*(S^V)$. We have (as G-modules)

$$C_i = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}[C_4/C_2] & 1 \leq i \leq 2k \\ \mathbb{Z}[C_4/e] & 2k+1 \leq i \leq 2k+2c \end{cases}$$

as is immediate from the cell structure in Section 2. We determine the differentials via the lemmas in section 3. We use the following bases

$$\begin{aligned} \mathbb{Z} &= \mathbb{Z}\{1\} \\ \mathbb{Z}[C_4/C_2] &= \mathbb{Z}\{e, \gamma\} \\ \mathbb{Z}[C_4/e] &= \mathbb{Z}\{e, \gamma, \gamma^2, \gamma^3\} \end{aligned}$$

- d_1: $\ker(d_0) = \mathbb{Z}$, so WLOG d_1 may be chosen to send $e \mapsto 1$, hence in the above bases we have
$$d_1 = \begin{pmatrix} 1 & 1 \end{pmatrix}$$
- d_i for $2 \leq i \leq 2k$, i even: in each case, we will have $\ker(d_{i-1}) = \mathbb{Z}\{e - \gamma\}$, so WLOG d_i may be chosen to send $e \mapsto e - \gamma$, hence in the above bases we have
$$d_i = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$
- d_i for $2 < i < 2k$, i odd: in each case, we will have $\ker(d_{i-1}) = \mathbb{Z}\{e + \gamma\}$, so WLOG d_i may be chosen to send $e \mapsto e + \gamma$, hence in the above bases we have
$$d_i = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

- d_{2k+1}: $\ker(d_{2k}) = \mathbb{Z}\{e+\gamma\}$, so WLOG d_{2k+1} may be chosen to send $e \mapsto e+\gamma$, hence in the above bases we have
$$d_{2k+1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$
- d_i for $2k+2 \leq i \leq 2k+2c$, i even: $\ker(d_{i-1}) = \mathbb{Z}\{e-\gamma, \gamma-\gamma^2, \gamma^2-\gamma^3\}$, so WLOG d_i may be chosen to send $e \mapsto e-\gamma$, hence in the above bases we have
$$d_i = \begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$
- d_i for $2k+2 < i < 2k+2c$, i odd: $\ker(d_{i-1}) = \mathbb{Z}\{e+\gamma+\gamma^2+\gamma^3\}$, so WLOG d_i may be chosen to send $e \mapsto e+\gamma+\gamma^2+\gamma^3$, hence in the above bases we have
$$d_i = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

We record this as follows:

$$\begin{tikzcd}
0 & 1 & 2 & 3 & \cdots & 2k-1 & 2k \\
\mathbb{Z} & \mathbb{Z}[C_4/C_2] \arrow[l, "\begin{pmatrix} 1 & 1 \end{pmatrix}"'] & \mathbb{Z}[C_4/C_2] \arrow[l, "\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}"'] & \mathbb{Z}[C_4/C_2] \arrow[l, "\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}"'] & \cdots \arrow[l] & \mathbb{Z}[C_4/C_2] \arrow[l] & \mathbb{Z}[C_4/C_2] \arrow[l, "\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}"']
\end{tikzcd}$$

$$\begin{tikzcd}
2k & 2k+1 & 2k+2 & 2k+3 & \cdots & 2k+2c-1 & 2k+2c \\
\mathbb{Z}[C_4/C_2] & \mathbb{Z}[C_4/e] \arrow[l, "\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}"'] & \mathbb{Z}[C_4/e] \arrow[l, "\begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}"'] & \mathbb{Z}[C_4/e] \arrow[l, "\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}"'] & \cdots \arrow[l] & \mathbb{Z}[C_4/e] \arrow[l] & \mathbb{Z}[C_4/e] \arrow[l, "\begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}"']
\end{tikzcd}$$

4.2. **The chain complex of fixed point Mackey functors.** We now apply the functor FP : $\mathbf{Mod}_{C_4} \to \mathbf{Mackey}_{C_4}$ to the above chain complex, which sends a C_4-module to its fixed point Mackey functor. We have only three C_4-modules in appearing in the chain complex, so we record each of their fixed point Mackey functors explicitly:

$$\mathbf{FP}(\mathbb{Z}) = \underline{\mathbb{Z}} = \begin{tikzcd}
\mathbb{Z} \arrow[d, bend right, "1"'] \\
\mathbb{Z} \arrow[u, bend right, "2"'] \arrow[d, bend right, "1"'] \\
\mathbb{Z} \arrow[u, bend right, "2"']
\end{tikzcd}$$

$$
\mathbf{FP}(\mathbb{Z}[C_4/C_2]) = \begin{tikzcd}
\mathbb{Z}\{e+\gamma\} \arrow[d, bend right] \\
\mathbb{Z}[C_4/C_2] \arrow[u, bend right] \arrow[d, bend right] \\
\mathbb{Z}[C_4/C_2] \arrow[u, bend right, "2"']
\end{tikzcd}
\qquad
\mathbf{FP}(\mathbb{Z}[C_4/e]) = \begin{tikzcd}
\mathbb{Z}\{e+\gamma+\gamma^2+\gamma^3\} \arrow[d, bend right] \\
\mathbb{Z}\{e+\gamma^2, \gamma+\gamma^3\} \arrow[u, bend right] \arrow[d, bend right] \\
\mathbb{Z}[C_4/e] \arrow[u, bend right]
\end{tikzcd}
$$

This gives:

$$
\begin{tikzcd}[column sep=small]
0 & 1 & 2 & 3 & \cdots & 2k-1 & 2k \\
\mathbb{Z} \arrow[d, bend right, "1"'] & \mathbb{Z}\{e+\gamma\} \arrow[l, "2"'] \arrow[d, bend right] & \mathbb{Z}\{e+\gamma\} \arrow[l, "0"'] \arrow[d, bend right] & \mathbb{Z}\{e+\gamma\} \arrow[l, "2"'] \arrow[d, bend right] & \cdots \arrow[l, "0"'] & \mathbb{Z}\{e+\gamma\} \arrow[l, "2"'] \arrow[d, bend right] & \mathbb{Z}\{e+\gamma\} \arrow[l, "0"'] \arrow[d, bend right] \\
\mathbb{Z} \arrow[u, bend right, "2"'] \arrow[d, bend right, "1"'] & \mathbb{Z}[C_4/C_2] \arrow[l, "\begin{pmatrix} 1 & 1 \end{pmatrix}"] \arrow[u, bend right] \arrow[d, bend right] & \mathbb{Z}[C_4/C_2] \arrow[l, "\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}"] \arrow[u, bend right] \arrow[d, bend right] & \mathbb{Z}[C_4/C_2] \arrow[l, "\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}"] \arrow[u, bend right] \arrow[d, bend right] & \cdots \arrow[l] & \mathbb{Z}[C_4/C_2] \arrow[l] \arrow[u, bend right] \arrow[d, bend right] & \mathbb{Z}[C_4/C_2] \arrow[l, "\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}"] \arrow[u, bend right] \arrow[d, bend right] \\
\mathbb{Z} \arrow[u, bend right, "2"'] & \mathbb{Z}[C_4/C_2] \arrow[l, "\begin{pmatrix} 1 & 1 \end{pmatrix}"] \arrow[u, bend right, "2"'] & \mathbb{Z}[C_4/C_2] \arrow[l, "\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}"] \arrow[u, bend right, "2"'] & \mathbb{Z}[C_4/C_2] \arrow[l, "\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}"] \arrow[u, bend right, "2"'] & \cdots \arrow[l] & \mathbb{Z}[C_4/C_2] \arrow[l] \arrow[u, bend right, "2"'] & \mathbb{Z}[C_4/C_2] \arrow[l, "\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}"] \arrow[u, bend right, "2"']
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
2k & 2k+1 & 2k+2 & 2k+3 & \cdots \\
\mathbb{Z}\{e+\gamma\} \arrow[d, bend right] & \mathbb{Z}\{e+\gamma+\gamma^2+\gamma^3\} \arrow[l, "4"'] \arrow[d, bend right] & \mathbb{Z}\{e+\gamma+\gamma^2+\gamma^3\} \arrow[l, "0"'] \arrow[d, bend right] & \mathbb{Z}\{e+\gamma+\gamma^2+\gamma^3\} \arrow[l, "4"'] \arrow[d, bend right] & \cdots \arrow[l, "0"'] \\
\mathbb{Z}[C_4/C_2] \arrow[u, bend right] \arrow[d, bend right] & \mathbb{Z}\{e+\gamma^2, \gamma+\gamma^3\} \arrow[l, "\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}"] \arrow[u, bend right] \arrow[d, bend right] & \mathbb{Z}\{e+\gamma^2, \gamma+\gamma^3\} \arrow[l, "\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}"] \arrow[u, bend right] \arrow[d, bend right] & \mathbb{Z}\{e+\gamma^2, \gamma+\gamma^3\} \arrow[l, "\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}"] \arrow[u, bend right] \arrow[d, bend right] & \cdots \arrow[l, "\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}"] \\
\mathbb{Z}[C_4/C_2] \arrow[u, bend right, "2"'] & \mathbb{Z}[C_4/e] \arrow[l, "\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}"] \arrow[u, bend right] & \mathbb{Z}[C_4/e] \arrow[l, "\begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}"] \arrow[u, bend right] & \mathbb{Z}[C_4/e] \arrow[l, "\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}"] \arrow[u, bend right] & \cdots \arrow[l, "\begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\cdots & 2k+2c-1 & 2k+2c \\
\cdots & \mathbb{Z}\{e+\gamma+\gamma^2+\gamma^3\} \arrow[l, "4"'] \arrow[d, bend right] & \mathbb{Z}\{e+\gamma+\gamma^2+\gamma^3\} \arrow[l, "0"'] \arrow[d, bend right] \\
\cdots & \mathbb{Z}\{e+\gamma^2,\gamma+\gamma^3\} \arrow[l, "{\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}}"] \arrow[u, bend right] \arrow[d, bend right] & \mathbb{Z}\{e+\gamma^2,\gamma+\gamma^3\} \arrow[l, "{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}"] \arrow[u, bend right] \arrow[d, bend right] \\
\cdots & \mathbb{Z}[C_4/e] \arrow[l, "{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}}"] \arrow[u, bend right] & \mathbb{Z}[C_4/e] \arrow[l, "{\begin{pmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}}"] \arrow[u, bend right]
\end{tikzcd}
$$

4.3. **The homology Mackey functors.** Taking homology, one has:

$$
\begin{tikzcd}
0 & 1 & 2 & 3 & \cdots & 2k-1 & 2k \\
\mathbb{Z}/2\{a_\sigma^b a_\lambda^c\} \arrow[d, bend right] & 0 \arrow[d, bend right] & \mathbb{Z}/2\{a_\sigma^{b-2} a_\lambda^c u_{2\sigma}\} \arrow[d, bend right] & 0 \arrow[d, bend right] & \cdots & 0 \arrow[d, bend right] & \mathbb{Z}/4\{a_\lambda^c u_{2\sigma}^k\} \arrow[d, bend right] \\
0 \arrow[u, bend right] \arrow[d, bend right] & 0 \arrow[u, bend right] \arrow[d, bend right] & 0 \arrow[u, bend right] \arrow[d, bend right] & 0 \arrow[u, bend right] \arrow[d, bend right] & \cdots & 0 \arrow[u, bend right] \arrow[d, bend right] & \mathbb{Z}/2\{a_{\sigma_2}^{2c}\} \arrow[u, bend right, "2"'] \arrow[d, bend right] \\
0 \arrow[u, bend right] & 0 \arrow[u, bend right] & 0 \arrow[u, bend right] & 0 \arrow[u, bend right] & \cdots & 0 \arrow[u, bend right] & 0 \arrow[u, bend right]
\end{tikzcd}
$$

$$
\begin{tikzcd}
2k & 2k+1 & 2k+2 & 2k+3 & \cdots & 2k+2c-1 & 2k+2c \\
\mathbb{Z}/4\{a_\lambda^c u_{2\sigma}^k\} \arrow[d, bend right] & 0 \arrow[d, bend right] & \mathbb{Z}/4\{u_{2\sigma}^k a_\lambda^{c-1} u_\lambda\} \arrow[d, bend right] & 0 \arrow[d, bend right] & \cdots & 0 \arrow[d, bend right] & \mathbb{Z}\{u_{2\sigma}^k u_\lambda^c\} \arrow[d, bend right] \\
\mathbb{Z}/2\{a_{\sigma_2}^{2c}\} \arrow[u, bend right, "2"'] \arrow[d, bend right] & 0 \arrow[u, bend right] \arrow[d, bend right] & \mathbb{Z}/2\{u_{2\sigma_2} a_\sigma^{2c-2}\} \arrow[u, bend right, "2"'] \arrow[d, bend right] & 0 \arrow[u, bend right] \arrow[d, bend right] & \cdots & 0 \arrow[u, bend right] \arrow[d, bend right] & \mathbb{Z}\{u_{2\sigma_2}^c\} \arrow[u, bend right, "2"'] \arrow[d, bend right] \\
0 \arrow[u, bend right] & 0 \arrow[u, bend right] & 0 \arrow[u, bend right] & 0 \arrow[u, bend right] & \cdots & 0 \arrow[u, bend right] & \mathbb{Z} \arrow[u, bend right, "2"']
\end{tikzcd}
$$

5. The non-orientable case

An actual C_4 representation $V = b\sigma + c\lambda$ is non-orientable if and only if b is odd, hence we assume for the remainder of the section that $b = 2k+1$ for $k \geq 0$.

5.1. **The chain complex of G-modules.** We use sections 2 and 3 to determine the chain complex of C_4-modules that determines $\underline{H}_*(S^V)$. We have (as G-modules)

$$
C_i = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}[C_4/C_2] & 1 \leq i \leq 2k+1 \\ \mathbb{Z}[C_4/e] & 2k+2 \leq i \leq 2k+1+2c \end{cases}
$$

as is immediate from the cell structure in Section 2. We determine the differentials via the lemmas in section 3. We use the following bases

$$\begin{aligned} \mathbb{Z} &= \mathbb{Z}\{1\} \\ \mathbb{Z}[C_4/C_2] &= \mathbb{Z}\{e, \gamma\} \\ \mathbb{Z}[C_4/e] &= \mathbb{Z}\{e, \gamma, \gamma^2, \gamma^3\} \end{aligned}$$

- d_1: $\ker(d_0) = \mathbb{Z}$, so WLOG d_1 may be chosen to send $e \mapsto 1$, hence in the above bases we have
$$d_1 = \begin{pmatrix} 1 & 1 \end{pmatrix}$$
- d_i for $2 \le i \le 2k$, i even: in each case, we will have $\ker(d_{i-1}) = \mathbb{Z}\{e - \gamma\}$, so WLOG d_i may be chosen to send $e \mapsto e - \gamma$, hence in the above bases we have
$$d_i = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$
- d_i for $2 < i \le 2k+1$, i odd: in each case, we will have $\ker(d_{i-1}) = \mathbb{Z}\{e+\gamma\}$, so WLOG d_i may be chosen to send $e \mapsto e + \gamma$, hence in the above bases we have
$$d_i = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$
- d_{2k+2}: $\ker(d_{2k+1}) = \mathbb{Z}\{e - \gamma\}$, so WLOG d_{2k+2} may be chosen to send $e \mapsto e - \gamma$, hence in the above bases we have
$$d_{2k+2} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$
- d_i for $2k+2 < i \le 2k+1+2c$, i odd: $\ker(d_{i-1}) = \mathbb{Z}\{e+\gamma, \gamma+\gamma^2, \gamma^2+\gamma^3\}$, so WLOG d_i may be chosen to send $e \mapsto e + \gamma$, hence in the above bases we have
$$d_i = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$
- d_i for $2k+4 \le i \le 2k+2c$, i even: $\ker(d_{i-1}) = \mathbb{Z}\{e - \gamma + \gamma^2 - \gamma^3\}$, so WLOG d_i may be chosen to send $e \mapsto e - \gamma + \gamma^2 - \gamma^3$, hence in the above bases we have
$$d_i = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}$$

We record this as follows:

$$\begin{tikzcd} 0 & 1 & 2 & 3 & \cdots & 2k & 2k+1 \\ \mathbb{Z} & \mathbb{Z}[C_4/C_2] \arrow[l, "\begin{pmatrix}1 & 1\end{pmatrix}"'] & \mathbb{Z}[C_4/C_2] \arrow[l, "\begin{pmatrix}1 & -1 \\ -1 & 1\end{pmatrix}"'] & \mathbb{Z}[C_4/C_2] \arrow[l, "\begin{pmatrix}1 & 1 \\ 1 & 1\end{pmatrix}"'] & \cdots \arrow[l] & \mathbb{Z}[C_4/C_2] \arrow[l] & \mathbb{Z}[C_4/C_2] \arrow[l, "\begin{pmatrix}1 & 1 \\ 1 & 1\end{pmatrix}"'] \end{tikzcd}$$

$$
\begin{tikzcd}
2k+1 & 2k+2 & 2k+3 & 2k+4 & \cdots & 2k+2c & 2k+1+2c \\
\mathbb{Z}[C_4/C_2] & \mathbb{Z}[C_4/e] \arrow[l, "{\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}}"'] & \mathbb{Z}[C_4/e] \arrow[l, "{\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}}"'] & \mathbb{Z}[C_4/e] \arrow[l, "{\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}}"'] & \cdots \arrow[l] & \mathbb{Z}[C_4/e] \arrow[l] & \mathbb{Z}[C_4/e] \arrow[l, "{\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}}"']
\end{tikzcd}
$$

5.2. **The chain complex of fixed point Mackey functors.** Applying $\mathbf{FP}(-)$ as above, we have

$$
\begin{tikzcd}
0 & 1 & 2 & 3 & \cdots & 2k & 2k+1 \\
\mathbb{Z} \arrow[d, bend right, "1"'] & \mathbb{Z}\{e+\gamma\} \arrow[l, "2"'] \arrow[d, bend right] & \mathbb{Z}\{e+\gamma\} \arrow[l, "0"'] \arrow[d, bend right] & \mathbb{Z}\{e+\gamma\} \arrow[l, "2"'] \arrow[d, bend right] & \cdots \arrow[l, "0"'] & \mathbb{Z}\{e+\gamma\} \arrow[l, "0"'] \arrow[d, bend right] & \mathbb{Z}\{e+\gamma\} \arrow[l, "2"'] \arrow[d, bend right] \\
\mathbb{Z} \arrow[u, bend right, "2"'] \arrow[d, bend right, "1"'] & \mathbb{Z}[C_4/C_2] \arrow[l, "{\begin{pmatrix} 1 & 1 \end{pmatrix}}"'] \arrow[u, bend right] \arrow[d, bend right] & \mathbb{Z}[C_4/C_2] \arrow[l, "{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}"'] \arrow[u, bend right] \arrow[d, bend right] & \mathbb{Z}[C_4/C_2] \arrow[l, "{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}"'] \arrow[u, bend right] \arrow[d, bend right] & \cdots \arrow[l] & \mathbb{Z}[C_4/C_2] \arrow[l] \arrow[u, bend right] \arrow[d, bend right] & \mathbb{Z}[C_4/C_2] \arrow[l, "{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}"'] \arrow[u, bend right] \arrow[d, bend right] \\
\mathbb{Z} \arrow[u, bend right, "2"'] & \mathbb{Z}[C_4/C_2] \arrow[l, "{\begin{pmatrix} 1 & 1 \end{pmatrix}}"'] \arrow[u, bend right, "2"'] & \mathbb{Z}[C_4/C_2] \arrow[l, "{\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}}"'] \arrow[u, bend right, "2"'] & \mathbb{Z}[C_4/C_2] \arrow[l, "{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}"'] \arrow[u, bend right, "2"'] & \cdots \arrow[l] & \mathbb{Z}[C_4/C_2] \arrow[l] \arrow[u, bend right, "2"'] & \mathbb{Z}[C_4/C_2] \arrow[l, "{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}"'] \arrow[u, bend right, "2"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
2k+1 & 2k+2 & 2k+3 & 2k+4 & \cdots \\
\mathbb{Z}\{e+\gamma\} \arrow[d, bend right] & \mathbb{Z}\{e+\gamma+\gamma^2+\gamma^3\} \arrow[l, "0"'] \arrow[d, bend right] & \mathbb{Z}\{e+\gamma+\gamma^2+\gamma^3\} \arrow[l, "2"'] \arrow[d, bend right] & \mathbb{Z}\{e+\gamma+\gamma^2+\gamma^3\} \arrow[l, "0"'] \arrow[d, bend right] & \cdots \arrow[l, "2"'] \\
\mathbb{Z}[C_4/C_2] \arrow[u, bend right] \arrow[d, bend right] & \mathbb{Z}\{e+\gamma^2, \gamma+\gamma^3\} \arrow[l, "{\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}}"'] \arrow[u, bend right] \arrow[d, bend right] & \mathbb{Z}\{e+\gamma^2, \gamma+\gamma^3\} \arrow[l, "{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}"'] \arrow[u, bend right] \arrow[d, bend right] & \mathbb{Z}\{e+\gamma^2, \gamma+\gamma^3\} \arrow[l, "{\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}}"'] \arrow[u, bend right] \arrow[d, bend right] & \cdots \arrow[l, "{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}"'] \\
\mathbb{Z}[C_4/C_2] \arrow[u, bend right, "2"'] & \mathbb{Z}[C_4/e] \arrow[l, "{\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}}"'] \arrow[u, bend right] & \mathbb{Z}[C_4/e] \arrow[l, "{\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}}"'] \arrow[u, bend right] & \mathbb{Z}[C_4/e] \arrow[l, "{\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}}"'] \arrow[u, bend right] & \cdots \arrow[l, "{\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=large]
\cdots & 2k+2c & 2k+2c+1 \\
\cdots & \mathbb{Z}\{e+\gamma+\gamma^2+\gamma^3\} \arrow[l, "0"'] \arrow[d, bend right] & \mathbb{Z}\{e+\gamma+\gamma^2+\gamma^3\} \arrow[l, "2"'] \arrow[d, bend right] \\
\cdots & \mathbb{Z}\{e+\gamma^2, \gamma+\gamma^3\} \arrow[l, "\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}"] \arrow[u, bend right] \arrow[d, bend right] & \mathbb{Z}\{e+\gamma^2, \gamma+\gamma^3\} \arrow[l, "\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}"] \arrow[u, bend right] \arrow[d, bend right] \\
\cdots & \mathbb{Z}[C_4/e] \arrow[l, "\begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{pmatrix}"] \arrow[u, bend right] & \mathbb{Z}[C_4/e] \arrow[l, "\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}"] \arrow[u, bend right]
\end{tikzcd}
$$

5.3. **The homology Mackey functors.** Taking homology, one has:

$$
\begin{tikzcd}
0 & 1 & 2 & 3 & \cdots & 2k & 2k+1 \\
\mathbb{Z}/2\{a_\sigma^b a_\lambda^c\} \arrow[d, bend right] & 0 \arrow[d, bend right] & \mathbb{Z}/2\{a_\sigma^{b-2} a_\lambda^c u_{2\sigma}\} \arrow[d, bend right] & 0 \arrow[d, bend right] & \cdots & \mathbb{Z}/2\{a_\sigma a_\lambda^c u_{2\sigma}^k\} \arrow[d, bend right] & 0 \arrow[d, bend right] \\
0 \arrow[u, bend right] \arrow[d, bend right] & 0 \arrow[u, bend right] \arrow[d, bend right] & 0 \arrow[u, bend right] \arrow[d, bend right] & 0 \arrow[u, bend right] \arrow[d, bend right] & \cdots & 0 \arrow[u, bend right] \arrow[d, bend right] & \mathbb{Z}/2\{a_{\sigma_2}^{2c}\} \arrow[u, bend right] \arrow[d, bend right] \\
0 \arrow[u, bend right] & 0 \arrow[u, bend right] & 0 \arrow[u, bend right] & 0 \arrow[u, bend right] & \cdots & 0 \arrow[u, bend right] & 0 \arrow[u, bend right]
\end{tikzcd}
$$

$$
\begin{tikzcd}
2k+2 & 2k+3 & 2k+4 & \cdots & 2k+2c \\
\mathbb{Z}/2\{a_\sigma u_{2\sigma}^k a_\lambda^{c-1} u_\lambda\} \arrow[d, bend right] & 0 \arrow[d, bend right] & \mathbb{Z}/2\{a_\sigma u_{2\sigma}^k u_\lambda^2 a_\lambda^{c-2}\} \arrow[d, bend right] & \cdots & \mathbb{Z}/2\{a_\sigma u_{2\sigma}^k u_\lambda^c\} \arrow[d, bend right] \\
0 \arrow[u, bend right] \arrow[d, bend right] & \mathbb{Z}/2\{u_{2\sigma_2} a_\sigma^{2c-2}\} \arrow[u, bend right] \arrow[d, bend right] & 0 \arrow[u, bend right] \arrow[d, bend right] & \cdots & 0 \arrow[u, bend right] \arrow[d, bend right] \\
0 \arrow[u, bend right] & 0 \arrow[u, bend right] & 0 \arrow[u, bend right] & \cdots & 0 \arrow[u, bend right]
\end{tikzcd}
$$

$$
\begin{tikzcd}
2k+1+2c \\
0 \arrow[d, bend right] \\
\mathbb{Z}/2\{u_{2\sigma_2}^c\} \arrow[u, bend right] \arrow[d, bend right] \\
0 \arrow[u, bend right]
\end{tikzcd}
$$

6. The positive cone

Putting it altogether we can describe the ring

$$\bigoplus_{a\in\mathbb{Z},b,c\geq 0} H_*(S^{a+b\sigma+c\lambda};\underline{\mathbb{Z}}) \cong \mathbb{Z}[u^{\pm}, a_\sigma, a_\lambda, u_{2\sigma}, u_\lambda]/(2a_\sigma, 4a_\lambda, a_\sigma^2 u_\lambda = 2a_\lambda u_{2\sigma})$$

where $u \in H_1(S^1;\underline{\mathbb{Z}}) \cong \mathbb{Z}$ is a generator. It's really more natural just to say that

$$\bigoplus_{b,c\geq 0} \pi^{C_4}_{*-b\sigma-c\lambda}(H\underline{\mathbb{Z}}) = \mathbb{Z}[a_\sigma, a_\lambda, u_{2\sigma}, u_\lambda]/(2a_\sigma, 4a_\lambda, a_\sigma^2 u_\lambda = 2a_\lambda u_{2\sigma})$$

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