Solution Set for Midterm
Math 266B, Winter 2007

1. For $u_x + x u_y = y$

$u(x,0) = 1$

The characteristics are

$$x_t = 1 \qquad x(0) = x_0$$
$$y_t = x \qquad y(0) = 0$$
$$u_t = y \qquad u(0) = 1$$

This has solution

$$x = x_0 + t \Rightarrow x_0 = x - t$$
$$y_t = x_0 + t \Rightarrow y = \tfrac{1}{2}t^2 + x_0 t = -\tfrac{1}{2}t^2 + xt$$
$$u_t = y \Rightarrow u = \tfrac{1}{6}t^3 + \tfrac{1}{2}x_0 t^2 + 1$$

Invert to get

$$x_0 = x - t$$
$$0 = \tfrac{1}{2}t^2 - xt + y \Rightarrow t = x \pm \sqrt{x^2 - 2y} \qquad \text{Use } - \text{ to get } t=0 \text{ on } y=0$$
$$= x - \sqrt{x^2 - 2y}$$

Then

$$u = \tfrac{1}{6}t^3 + \tfrac{1}{2}(x-t)t^2 + 1$$
$$= -\tfrac{1}{3}t^3 + \tfrac{1}{2}xt^2 + 1$$
$$\boxed{u = -\tfrac{1}{3}\left(x - \sqrt{x^2-2y}\right)^3 + \tfrac{1}{2}x\left(x - \sqrt{x^2-2y}\right)^2 + 1}$$

2. ~~The characteristics are~~

Set ~~$F(t,x,u,p=u_t,q=$~~

$$F(t, x, u, p = u_t, q = u_x) = p + uq^2 - xu^2$$

The characteristics are then

$$t_s = F_p = 1$$

$$x_s = F_q = 2uq$$

$$u_s = pF_p + qF_q = p + 2uq^2$$

$$p_s = -F_t - F_u p = -(q^2 - 2xu)p$$

$$q_s = -F_x - F_u q = u^2 - (q^2 - 2xu)q$$

3. The characteristics are

C_u: $x_t = 1$ for u, i.e. u constant on $x - t =$ const

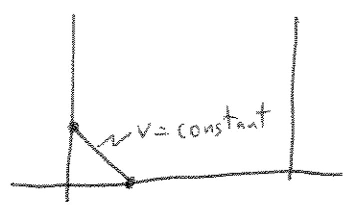
C_v: $x_t = -1$ for v $\quad$ v constant on $x + t =$ const.

The characteristic C_u is incoming on $x = 0$

The characteristic C_v is incoming on $x = 1$

(a) is well-posed since data is prescribed for incoming characteristics.

(b) is not well-posed. The data is prescribed on outgoing characteristics. For example v on $x=0$ is found from initial data.



(c) is well-posed. From initial data v is determined on $x=0$ and u on $x=1$, ~~from the bdry~~ up to time 1. Then from bdry data, u is determined on $x=0$ and v on $x=1$. ~~Then~~ This is repeated (i.e. v is determined on $x=0$ and u on $x=1$, for $1 < t < 2$ from the opposite bdry.), to get the solution for all (x,t).

4. (i) Kirchoff's formula is

$$u(x,t) = \frac{1}{4\pi}\partial_t\left(t\int_{|\xi|=1} g(x+ct\xi)\,dS_\xi\right) + \frac{t}{4\pi}\int_{|\xi|=1} h(x+ct\xi)\,dS_\xi$$

For $x=0$, the argument of g and h is $ct\xi$ with $|ct\xi|=ct$. Since g and h are spherically symmetric, then

$$g(ct\xi) = g(ct)$$
$$h(ct\xi) = h(ct)$$

Moreover $\int_{|\xi|=1} dS_\xi = 4\pi$. Thus

$$u(0,t) = \partial_t\big(t g(ct)\big) + t h(ct)$$

(ii) For $h=0$

$$g = \begin{cases} 0 & r<1 \\ r-1 & r>1 \end{cases}$$

then

$$u(0,t) = g(ct) + ct\,g'(ct)$$

$$= \begin{cases} 0 & ct<1 \\ (ct-1) + ct & ct>1 \end{cases}$$

This has values

$$u(0, t=c^{-1}-) = 0$$
$$u(0, t=c^{-1}+) = 1$$

with a discontinuity at $t=c^{-1}$.