ADE Math 266
Final Solutions
Winter 2007

1. (a) G satisfies

$$\partial_t \hat{G} = -k^4 \hat{G} \qquad t \geq 0$$
$$\hat{G} = \frac{1}{2\pi} \qquad t = 0$$

so that

$$\hat{G}(k,t) = \frac{1}{2\pi} e^{-k^4 t}$$
$$G(x,t) = \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-k^4 t} e^{ikx}\, dk$$

For $t>0$, this is convergent, since e^{-tk^4} decays rapidly as $|k| \to \infty$.

(b) The solution is

$$u(x,t) = \int_{-\infty}^{\infty} G(x-y,t)\, u_0(y)\, dy$$

(c) For small t, and small x,

$$u(x,t) \cong u(x,0) + t\, u_t(x,0)$$
$$= u_0(x) - t\, u_{xxxx}$$
$$\cong x^4 - t\,4!$$
$$< 0 \quad \text{if } x=0,\ t>0$$

(d) ~~If u satisfies~~ For the example in (c),

$-u(x,t) > 0$ for some x, $t>0$

and $-u(x,0) < 0$.

This violates max principle, since $-u$ solve the eqtn

So no max principle.

2. (a) Look for $u = \psi_x$, then (3) becomes

$$\psi_{xt} + \psi_x \psi_{xx} = \nu \psi_{xxx}$$

Integrate once to get

$$\psi_t + \tfrac{1}{2}\psi_x^2 = \nu \psi_{xx}$$

with constant set to 0, since we ~~expect~~ can absorb constants into ψ_t.

Now set $\psi = \alpha \log \theta$

$$\psi_t = \alpha\, \theta_t / \theta$$

$$\psi_x = \alpha\, \theta_x / \theta$$

$$\psi_{xx} = \alpha\, \theta_{xx}/\theta - \alpha\, \theta_x^2/\theta^2$$

Choose $\alpha = -2\nu$ so that

$$\tfrac{1}{2}\psi_x^2 = \frac{4\nu^2}{2}\,\frac{\theta_x^2}{\theta^2}$$

$$= -\nu\alpha\, \theta_x^2/\theta^2$$

This eliminates the nonlinear terms in the equation, so that the ψ equation become

$$\alpha\, \theta_t/\theta = \nu\alpha\, \theta_{xx}/\theta$$

i.e.

$$\theta_t = \nu\, \theta_{xx}$$

(b) Since $u > 0$, then $(\log\theta)_x > 0$ and so $\theta_x > 0$, at $t = 0$.

By max principle, $\theta_x > 0 \quad \forall t$.

So $(\log \theta)_x > 0 \quad \forall t$

and so $u > 0 \quad \forall t$.

3. The solution is (for polar coordinates (r, θ))

$$u = 1 - 2\theta/\pi$$

Then

$$u_y = \begin{cases} r^{-1} u_\theta \\ -r^{-1} u_\theta \end{cases} = \begin{cases} -\dfrac{2}{\pi r} < 0 & x > 0 \\ 2/\pi r > 0 & x < 0 \end{cases}$$

Let u solve $\Delta u = f$. ~~Under~~ $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. For any δu

4.(a) $$I(u+\delta u) = \int_\Omega \tfrac{1}{2}|\nabla(u+\delta u)|^2 + (u+\delta u) f\,dx$$
$$= \int_\Omega \tfrac{1}{2}|\nabla u|^2 + \nabla u\cdot\nabla\delta u + |\nabla \delta u|^2 + uf + \delta u f\,dx$$
$$= \int_\Omega \left(\tfrac{1}{2}|\nabla u|^2 + uf\right)dx + \int_\Omega |\nabla\delta u|^2 dx + \int_\Omega \nabla u\cdot\nabla\delta u + \delta u f\,dx$$
$$= I(u) + \int_\Omega |\nabla\delta u|^2 dx + \int_\Omega -\delta u(\nabla^2 u - f)\,dx$$
$$= I(u) + \int_\Omega |\nabla\delta u|^2 dx \qquad \text{since } \nabla^2 u = f$$
$$< I(u) \quad \text{unless } \delta u \equiv 0$$

In the integration, there are no bdry terms, since $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$

(b) Let u solve $\Delta u = f$, $u = 0$ on $\partial\Omega$. For any δu with $\delta u = 0$ on $\partial\Omega$, the same calculation works. The bdry terms are
$$\int_{\partial\Omega} \delta u \frac{\partial u}{\partial n}\,ds = 0$$
since $\delta u = 0$ on $\partial\Omega$.