Final Exam
APPLIED DIFFERENTIAL EQUATIONS 266B
Winter 2007
Due: 11am, Thursday 3/22/07
Total time allowed for exam: 8 hours

1. Consider the PDE

$$\begin{aligned} u_t &= -u_{xxxx} \quad \text{for} \quad -\infty < x < \infty,\ t \geq 0 \\ u(x,0) &= u_0(x) \quad \text{for} \quad -\infty < x < \infty. \end{aligned} \tag{1}$$

   (a) Find the Green's function $G(x,t)$ satisfying

$$\begin{aligned} G_t &= -G_{xxxx} \quad \text{for} \quad -\infty < x < \infty,\ t \geq 0 \\ G(0,x) &= \delta(x). \end{aligned} \tag{2}$$

   A Fourier integral for G is sufficient, but show that the integral is convergent.

   (b) Show that (1) is solveable by finding an integral formula for its solution using G.

   (c) Consider the solution of (1) with $u_0(x) = x^4 e^{-x^2}$. In particular, show that $u(0,t) < 0$ for $0 < t << 1$, by performing a Taylor expansion around $(x,t) = (0,0)$.

   (d) Show that (1) does not satisfy a maximum principle.

2. Consider the equation

$$u_t + uu_x = \nu u_{xx}. \tag{3}$$

   (a) Show that the transformation $u = -2\nu(\log\theta)_x = -2\nu\theta_x/\theta$ reduces (3) to the heat equation

$$\theta_t = \nu\theta_{xx}. \tag{4}$$

   (b) Assume that u is uniformly bounded and positive for $t = 0$; i.e., for some constants a_1 and a_2,

$$0 < a_1 < u(x,0) < a_2 < \infty \tag{5}$$

   for all x. Also assume that

$$u \to u_\pm \text{ as } x \to \pm\infty \tag{6}$$

   for all $t \geq 0$, with $u_\pm > 0$. Show that u remains uniformly bounded and positive for any finite time $t > 0$.

3. For the Dirichlet problem on a half-space

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \text{ for } -\infty < x < \infty,\ y \geq 0 \\ u(x,0) &= -1 \text{ for } -\infty < x < 0 \\ u(x,0) &= 1 \text{ for } 0 < x < \infty \end{aligned} \tag{7}$$

show that

$$\begin{aligned} u_y(x,0) &> 0 \text{ for } -\infty < x < 0 \\ u_y(x,0) &< 0 \text{ for } 0 < x < \infty. \end{aligned} \tag{8}$$

4. For a bounded open set Ω in R^n and a given function f, consider the integral

$$I(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + uf dx. \tag{9}$$

(a) Show that the minimzer of I, over all smooth functions u, is the solution $u = u_N$ of the Neumann problem

$$\begin{aligned} \Delta u &= f \text{ for } x \in \Omega \\ \partial u / \partial n &= 0 \text{ for } x \in \partial\Omega. \end{aligned} \tag{10}$$

Hint: Show that $0 < I(u + \delta u) - I(u)$ for all functions $\delta u \neq 0$.

(b) If I is minimized over all smooth function u with $u = 0$ on $\partial\Omega$, show that the miminimizer is the solution $u = u_D$ of the Dirichlet problem

$$\begin{aligned} \Delta u &= f \text{ for } x \in \Omega \\ u &= 0 \text{ for } x \in \partial\Omega. \end{aligned} \tag{11}$$

Use the same hint, but with $u = \delta u = 0$ on $\partial\Omega$.