HW 6 Solution Set

p. 110 #8. Suppose that $\Delta u \geq 0$, u is not constant and

$u(x_0) = \sup\{u(x) : x \in \Omega\}$. Let $B_\varepsilon(x_1)$ be ball inside Ω, touching

x_0 on $\partial\Omega$, as in the assumption. By max principle, $u < u(x_0)$ in Ω.

So $u < u(x_0)$ on $\partial B_\varepsilon(x_1)$ except at $x = x_0$. Set

$$w = \begin{cases} u(x_0) & \text{on the half sphere } S^+ \text{ on } \partial B_\varepsilon(x_1) \text{ centered at } x_0 \\ \max\limits_{\partial B_\varepsilon - S^+} u(x) < u(x_0) & \end{cases}$$

w harmonic in $B_\varepsilon(x_1)$

From ~~Green~~ Poisson formula, one can check that $\frac{\partial w}{\partial \nu} > 0$ at x_0.

Also $u \leq w$ by max principle for subharmonic function $u - w$ on $B_\varepsilon(x_1)$.

So

$$\begin{aligned} \frac{\partial u}{\partial \nu} &\cong u(x_0) - u(x_0 - \delta n) \\ &\geq w(x_0) - w(x_0 - \delta n) \qquad \text{since } w(x_0) = u(x_0) \\ &\qquad\qquad\qquad\qquad\qquad\quad w(x_0 - \delta n) > u(x_0 - \delta n) \\ &\cong \frac{\partial w}{\partial n} \\ &> 0. \end{aligned}$$

#9 The proofs are the same as for a function satisfying $\Delta u = 0$.

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\# 3. (a) For $\mathbb{R}^n_+$, $G(x,\xi) = K(x-\xi) - K(x-\xi^*)$

$$= K(x-\xi) - K(x_1-\xi_1, \ldots, x_n-\xi_n, x_n+\xi_n)$$

Each of these is symmetric

For $B_a(0)$, $G(x,\xi) = K(x-\xi) - \left(\frac{a}{|\xi|}\right)^{n-2} K(x-\xi^*)$ for $n>2$.

The first term is symmetric.

The second term is

$$\left(\frac{a}{|\xi|}\right)^{n-2} \left|x - a^2\xi/|\xi|^2\right|^{-(n-2)} = \left(\frac{|\xi|^2}{a^2}\left|x-a^2\xi/|\xi|^2\right|^2\right)^{-\frac{n-2}{2}}$$

$$= \left(a^{-2}\,|\xi|^2\left(|x|^2 - 2a^2\, x\cdot\xi/|\xi|^2 + a^4|\xi|^{-2}\right)\right)^{-\frac{n-2}{2}}$$

$$= \left(a^{-2}\left(|\xi|^2|x|^2 - 2a^2\, x\cdot\xi + a^4\right)\right)^{-\frac{n-2}{2}}$$

which is symmetric (i.e. unchanged by switching x and ξ).

(b) The Green's function can be used to solve the inhomogeneous problem; i.e.

$$u(x) = \int_\Omega G(x,\xi) f(\xi)\, d\xi$$

solves

$$\Delta u = f \quad \text{in } \Omega$$
$$u = 0 \quad \text{on } \partial\Omega$$

Now let u, v be two solutions, with forcing terms f and g.

Since $u = v = 0$ on $\partial\Omega$, the

$$\int_\Omega u\,\Delta v\, dx = \int_\Omega v\,\Delta u\, dx$$

and $u(x) = \int_\Omega G(x,\xi) f(\xi)\, d\xi$
$v(x) = \int_\Omega G(x,\xi) g(\xi)\, d\xi$

i.e.

$$\int_\Omega u g\, dx = \int_\Omega v f\, dx$$

i.e.

$$\int_\Omega\int_\Omega G(x,\xi) f(\xi) g(x)\, d\xi\, dx = \int_\Omega\int_\Omega G(x,\xi) g(\xi) f(x)\, d\xi\, dx$$

$$= \int_\Omega\int_\Omega G(\xi,x) f(\xi) g(x)\, d\xi\, dx$$

for all f, g which implies $G(x,\xi) = G(\xi,x)$.

4. (a) Let $\Delta u = f$ in Ω

$u = 0$ $\partial\Omega$

with $f \geq 0$.

The solution is $u(x) = \int G(x,\xi) f(\xi) d\xi$.

From the weak max principle $u(x) \leq \max_{x' \in \partial\Omega} u(x') = 0$.

Since this is for any $f \geq 0$, then

$$G(x,\xi) \leq 0.$$

(b) Proof works the same, using $f > 0$ ~~for~~ in Ω.

6. The Green's function in the quarter plane $\{(x,y) : x > 0, y > 0\}$ is

$$G(\underset{\sim}{x}, \underset{\sim}{\xi}) = K(\underset{\sim}{x} - \underset{\sim}{\xi}) - K(\underset{\sim}{x} - R_1 \underset{\sim}{\xi}) - K(\underset{\sim}{x} - R_2 \underset{\sim}{\xi}) + K(\underset{\sim}{x} - R_1 R_2 \underset{\sim}{\xi})$$

in which

$$R_1(\xi_1, \xi_2) = (-\xi_1, \xi_2)$$
$$R_2(\xi_1, \xi_2) = (\xi_1, -\xi_2)$$
$$R_1 R_2(\xi_1, \xi_2) = (-\xi_1, -\xi_2)$$