Math 181: Final Exam

March 22, 2007

NOTATION.
For an option, T is exercise time (in years) and X is strike price. The risk-free interest rate is r. For the Black-Scholes model, S is the underlying stock price; t is time; σ and μ are the volatility and average growth rate of S. Always assume that $\mu > r$.
For the binary random walk model, S_n is the stock price after n steps, u and d are the factors for increase and decrease of the equity price, dt is the time step, the real probability for an up step is p', and the risk-neutral probability for an up step is p.

1. Consider a put option, for an equity following the Black-Scholes model, with $T = 1$ and $X = 100$ on an equity with initial price $S(0) = 110$, and with $\sigma = 0.1$, $\mu = 0.1$ and $r = 0.05$.

(a) What is the value $p(0)$ of the put option at $t = 0$?

(b) What is the value of Δ for this option at $t = 0$?

2. Consider $f_2(S,t) = S^2 exp\{(r + \sigma^2)(T - t)\}$.

(a) Show that f_2 solves the Black-Scholes equation.

(b) What is the payout at expiration for this option?

(c) Find a portfolio $\Pi = f_2 + \alpha S$ consisting of a single unit of this option and α units of the underlying stock S, chosen so that the portfolio is Δ-neutral at $t = 0$; i.e., find the value of α.

(d) Find a version of put-call parity for a call c_2 with payout $\max(S^2 - X, 0)$ and put p_2 with payout $\max(X - S^2, 0)$ at $t = T$.

3. For a binary random walk (i.e., a tree or CRR) model of a stock, we found that

$$\begin{aligned} u &= e^{\sigma\sqrt{dt}} \qquad d = e^{-\sigma\sqrt{dt}} \\ p &= (e^{r dt} - d)/(u - d) \end{aligned} \tag{1}$$

if the time step is dt. Similarly for the real probability p' for an up step, one can use

$$p' = (e^{\mu dt} - d)/(u - d). \tag{2}$$

(a) Find a constraint on the parameters of the model so that the probabilities make sense; i.e., so that p and p' are between 0 and 1.

(b) If the constraint of (a) is violated, which of the parameters in the model (μ, σ, r, dt) should be changed? Please justify your answer. Hint: Some of the parameters come from market data; while others are a modeling choice.

4. A 1 year American put option on a non-dividend-paying stock has a $X = \$102$, $S_0 = \$100$, $r = .05$. Use a one-step binomial tree with $u = 1.1$ and $d = .9$ to calculate the American put option price P. Be sure to state whether early exercise is applied.