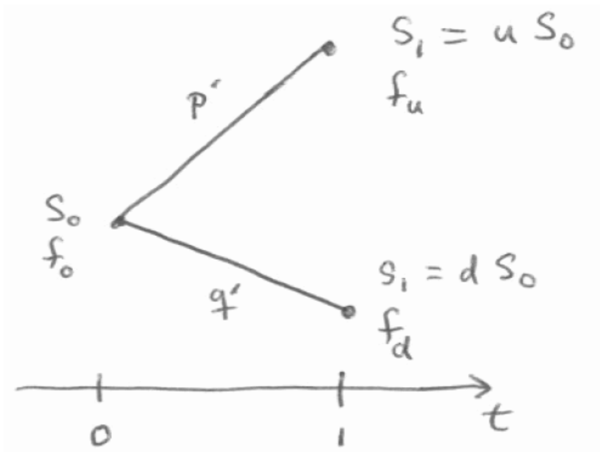


Figure 1: A single step of a binary random walk, with stock price $S$, option price $f$ and probabilities $p'$ and $q'$ of going up ($u$) or down ($d$).

# Math 181 Lecture 9

## Pricing Options for the Random Walk with Discrete Steps

Consider a single step in a random walk

$$S_0 \text{ given at } t = 0$$
$$S_1 = \begin{cases} uS_0 & \text{probability p'} \\ dS_0 & \text{probability q' = 1-p'} \end{cases}.$$

Also consider an option where value is

$$\begin{array}{ll} f_0 & \text{at } t = 0 \\ f_u & \text{at } t = 1, \text{ if step is up} \\ f_d & \text{at } t = 1, \text{ if step is down.} \end{array}$$

This is summarized in in Figure 1.

For example, if the option is a call with a strike price $X$ such that $dS_0 < X < uS_0$, and if $t_1$ is the expiration time, then

$$\begin{aligned} f_u &= uS_0 - X \\ f_d &= 0. \end{aligned}$$

It is tempting to think that the option price $f_0$ at $t_0$ is the expected value of the future option price, suitably discounted. Discounting the expected value $f_1$ back to $t = 0$ gives

$$\tilde{f}_0 = e^{-rdt} E(f_1) = e^{-rdt}(p' f_u + (1 - p') f_d). \tag{1}$$

As shown below, however, this price for $f$ would not be arbitrage free!

In order to correctly price $f$, we perform an arbitrage-free analysis by considering two portfolios

**A:** long $\Delta$ shares of stock, each with value $S_0$; short one option of value $f_0$

**B:** cash $K$

at time $t_0$. We will choose $\Delta$ so that there is no risk in the portfolio A. Then we'll choose $K$ to make $B$ equivalent to $A$. The value of $A$ at $t_0$ and $t_1$ is

$$\begin{aligned} A(t = t_0 = 0) &= \Delta S_0 - f_0 \\ A(t = t_1 = dt) &= \begin{cases} \Delta u S_0 - f_u & \text{if up} \\ \Delta d S_0 - f_d & \text{if down} \end{cases}. \end{aligned}$$

The profile will be riskless if the two outcomes are identical, i.e. if

$$\Delta u S_0 - f_u = \Delta d S_0 - f_d$$

i.e. if

$$\Delta = \frac{f_u - f_d}{uS_0 - dS_0}. \tag{2}$$

Now for portfolio B, take an amount of cash $K$ equal to the initial value of $A$, i.e. at $t = t_0$,

$$B(t_0) = K = \Delta S_0 - f_0.$$

Then at $t = t_1$

$$B(t_1) = K e^{rdt}.$$

Since portfolio A is riskless and its initial value is the same as B's, then its value at $t_1$ is also the same; otherwise there would be an arbitrage opportunity. This shows that

$$\begin{aligned} A(t_1) &= B(t_1) \\ &= e^{rdt} B(t_0) \\ &= e^{rdt} A(t_0). \end{aligned}$$

In the future, we will dispense with the second portfolio. We will summarize this argument by saying that since $A$ is riskfree, its value must growth at the riskfree rate $r$.

Now insert the values of $A(t_0),\ A(t_1)$ and $\Delta$ to obtain

$$(\Delta u S_0 - f_u) = e^{rdt}(\Delta S_0 - f_0)$$

which leads to

$$f_0 = e^{-rdt}(pf_u + (1-p)f_d) \tag{3}$$

in which

$$p = \frac{e^{rdt} - d}{u - d}. \tag{4}$$

The formula (3) is a simplified version of the Black-Scholes equation for pricing as option. It has some surprising properties:

**(i)** The formula (3) for $f_0$ looks like a discounted average, as in (1). But the probability $p$ in (3) is not the probability $p'$ in for the the random walk to move up. In fact $p$ is determined just from $u, d, r$ and $dt$.

**(ii)** The formula is the same no matter what type of option is involved. It could be a call or put. The values of $f_u$ and $f_d$ would be different, but the relation between $f_0$ and these values would be the same.

**(iii)** The probabilities $p'$ and $q'$, for the stock to move up or down, do not influence the option price! They do not appear in (3) or (4). This says that the likelihood of the stock to move up or down does not affect the value of the option.

Although this is surprising, it has an explanation. The argument above shows that the risk in the option value is entirely due to risk in the stock, since a portfolio of stock and options can be riskless. Since the risks in the stock are already accounted for in its price, they do not enter into the price of the option.

This pricing method for the option is called "risk-neutral valuation". Further understanding of this comes from relating the model parameters $p$, $u$ and $d$ to financial parameters $\mu$ and $\sigma$.

By the definition of $\mu$, the average value at $t = t_1 = dt$ should be $e^{\mu dt} S_0$; i.e.,

$$\begin{aligned} e^{\mu dt} S_0 &= E[S_1] \\ &= p' u S_0 + (1 - p') d S_0 \end{aligned}$$

which implies

$$p' = \frac{e^{\mu dt} - d}{u - d}. \tag{5}$$

By the definition of $\sigma$, the variance in the value of $S_1$ should (approximately) equal $\sigma^2 dt S_0^2$; i.e.,

$$\begin{aligned} \sigma^2 dt S_0^2 &= E[(S_1 - E[S_1])^2] \\ &= (p'u^2 + (1-p')d^2 - (p'u + (1-p')d)^2) S_0^2. \end{aligned} \tag{6}$$

We also assume that $d = u^{-1}$. The solution of (6) with an error of size $dt^{3/2}$ is

$$\begin{aligned} u &= e^{\sigma\sqrt{dt}} \\ d &= e^{-\sigma\sqrt{dt}}. \end{aligned} \tag{7}$$

The pricing formula (3) is exactly the same as the naive average (1), but using $p$ instead of $p'$. Comparison of (4) and (5) shows that the probability $p$ used in the pricing formula (3) is the same as the formula (4) for the real probability $p'$ if the growth rate $\mu$ were equal to the risk-free rate $r$. The growth rate $\mu$ equals $r$ if there is no risk-premium, which would only be true if investors have a risk-neutral view; i.e., if they do not require additional return to compensate for taking risk. Although this is not the case for the real price process for $S$, it does give a correct method for pricing the option $f$, as a function of $S$. This is the risk-neutral pricing method.

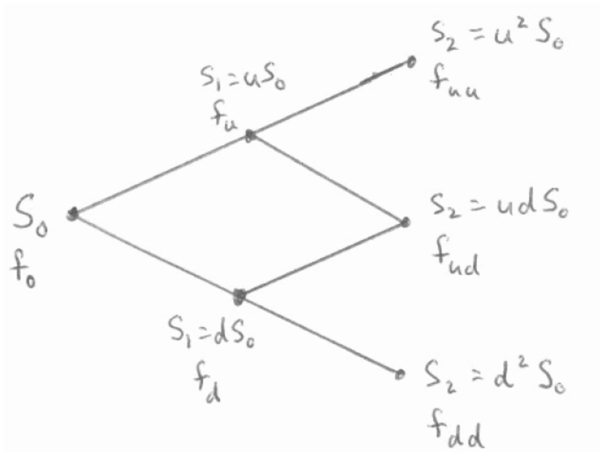


Figure 2: Two steps of a binary random walk, with stock price $S$, option price $f$ and probabilities $p'$ and $q'$ of going up ($u$) or down ($d$).

# Math 181 Lecture 10

## Pricing Options on a Binomial Tree

The price of an option on a binomial tree can be determined from its values at expiration, by repeated application of the single step formulas from Lecture 9. First we perform this for a 2 step tree, then we generalize to an arbitrary sized tree.

Consider the two step tree in Figure 2.

In this formulation, the values $S_0, u, d$ are known. They are the initial stock value, and the proportional increase or decrease in its value. We take the final time to be the exercise time for the option; so that $f_{uu}, f_{ud}, f_{dd}$ are also known. (Note that $f_{du} = f_{ud}$.) We wish to compute $f$, and along the way we will compute $f_u$ and $f_d$.

Computation of $f_u$ and $f_d$ is performed on one step trees, using the values of $f_{uu}$, $f_{ud}$ and $f_{dd}$, as shown in Figure 3.

The value $f_u$ is computed as a suitable discounted average of $f_{uu}$ and $f_{du}$ as in (3), (4) by the substitution $(f_{uu}, f_{du}, f_n)$ for $(f_u, f_d, f_0)$. The result is

$$f_u \quad = \quad e^{-rdt}(pf_{uu} + (1-p)f_{du})$$

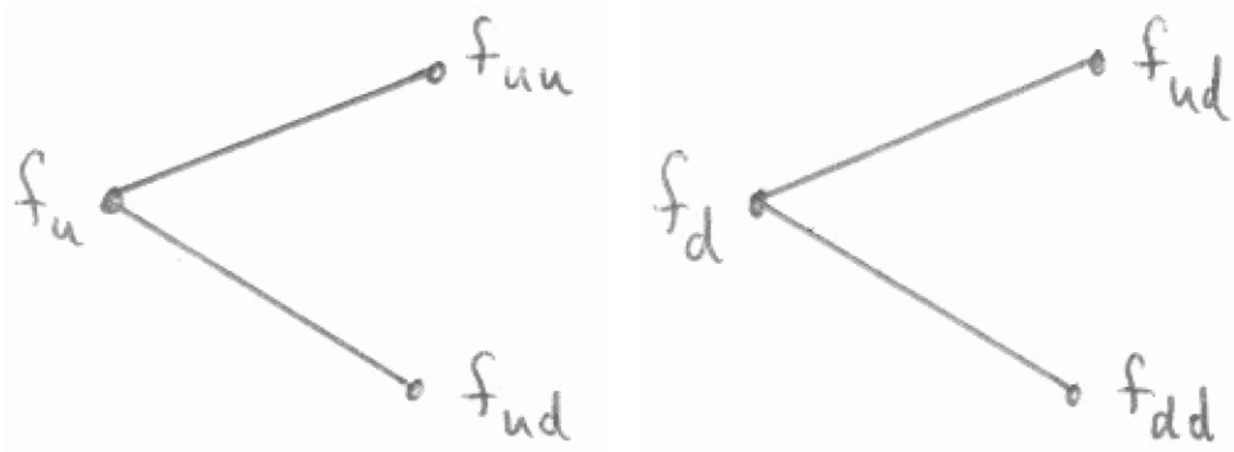


Figure 3: One step trees used for computation of $f_u$ (left) and $f_d$ (right).

$$p = \frac{e^{rdt} - d}{u - d}.$$

In a similar way

$$f_d = e^{-rdt}(pf_{ud} + (1-p)f_{dd})$$

with the same $p$.

Now we compute $f_0$ using the tree shown in Figure 4.

$$\begin{aligned} f_0 &= e^{-rdt}(pf_u + (1-p)f_d) \\ &= e^{-r2dt}\{p(pf_{uu} + (1-p)f_{ud}) + (1-p)(pf_{ud} + (1-p)f_{dd})\} \\ &= e^{-r2dt}\left\{p^2 f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 f_{dd}\right\}. \end{aligned}$$

Note that

$$\begin{aligned} p^2 + 2p(1-p) + (1-p)^2 &= (p + (1-p))^2 \\ &= 1 \end{aligned}$$

so that this formula is still a discounted average.

## Risk-Neutral Valuation

We now give a tree-interpretation of these formulas. The real random walk has probabilities $p', q' = (1-p')$ of stepping up or down. We replace these

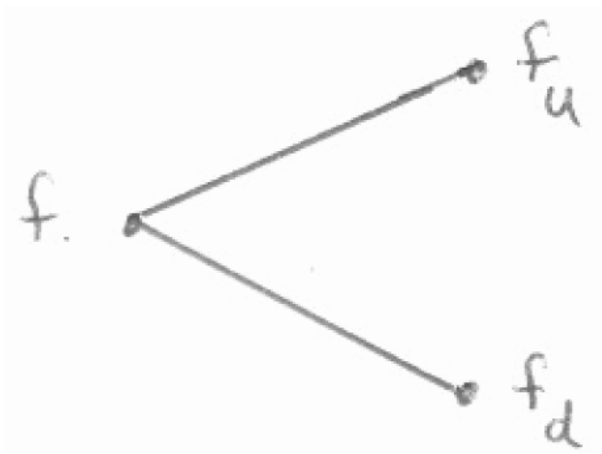


Figure 4: One step tree used for computation of $f$ from $f_u$ and $f_d$.

probabilities with the probabilities $p$ and $q = 1 - p$ that occur in the option pricing formula, as shown in Figure 5.

The resulting random walk is called the "risk-neutral process". This is not the real process. It has the same states $S_0, uS_0, dS_0$, etc., but they occur with different probabilities.

The usefulness of the risk-neutral process is that it can be directly used to calculate the option price as an average (with discounting). For a single step the tree is drawn in Figure 6 and the price is

$$\begin{aligned} f_0 &= e^{-rdt} \bar{E}(f_1) \\ &= e^{-rdt}(pf_u + (1-p)f_d) \end{aligned}$$

in which $\bar{E}$ denotes average with respect to the risk-neutral process.

For the two step process the same formulas work:

$$\begin{aligned} f_0 &= e^{-r2dt} \bar{E}(f_2) \\ &= e^{-r2dt}(p^2 f_{uu} + 2p(1-p)f_{ud} + (1-p)^2 f_{dd}). \end{aligned}$$

More generally its true that for $m < n$

$$f_m = e^{-r(t_n - t_m)} \bar{E}(f_n). \tag{8}$$

This is the risk-neutral valuation formula.

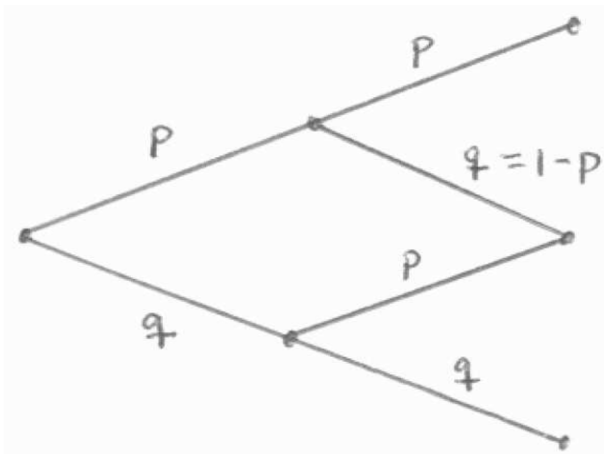


Figure 5: Two steps of a tree with risk-neutral probabilities $p$ and $q = 1 - p$.

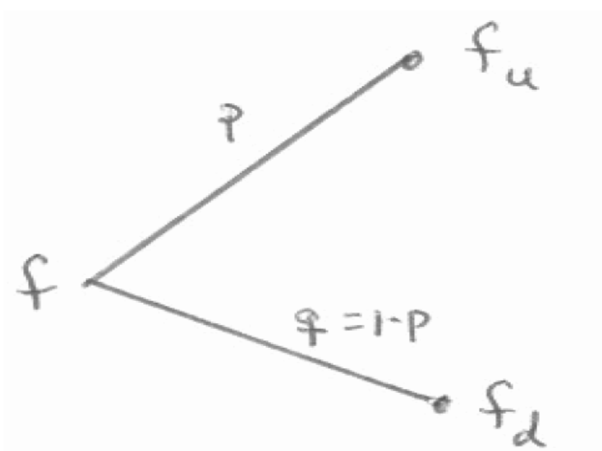


Figure 6: One step tree, with risk-neutral probabilities used for computation of $f_0$ from $f_u$ and $f_d$.