**Math 181** **Lecture 6**

## Financial Derivatives

Financial derivatives are securities whose values are based on the values of other securities. Here are some examples

- forward contracts
- futures contracts
- call option
- put options
- interest rate swaps
- American options

These are described in Hull's book (Chapter 1 for forwards, futures, calls and puts; Chapter 5 for interest rate swaps). The discussion is not repeated here, but will be presented in class.

Some important terminology is "long" and "short". In every transaction, there is a buyer (the long position) and a seller (the short position). Again most discussion is contained in Hull's Chapter 1. Short-selling is also allowed.

Hull also describes 3 kinds of users of derivative securities:

**(i)** hedgers, who use derivatives to reduce risk that they face through other investments or business

**(ii)** speculators, who take financial risks in order to profit from their expectation of the financial future

**(iii)** arbitrageurs, who use derivatives to obtain riskless profits. They do this by constructing two financial positions, which are equivalent, but which are priced differently. Examples of arbitrage will be discussed in the next lecture.

Finally, we will frequently use a basic interest rate $r$, called the risk-free rate of return. This is the rate that can be obtained with no risk of loss. An example would be the overnight rate charged by the Federal Reserve. For most of our purposes this will be assumed to be constant in time.

Note that $r$ is not the rate that you can get as an individual. Banks lower the rates for savings accounts as a sort of service fee.

We also assume continuous compounding. This means that an amount $x_0$ at time $t = 0$, will be worth

$$x(t) = x_0 e^{rt}$$

at time $t$. Time $t$ will be measured in years and $r$ in inverse years (i.e. rate per year) here.

The reverse of this is that a cash payment $x_T$ at time $T$ in the future has value

$$x_0(0) = x_T e^{-rT}$$

at time 0, or more generally at time $t$

$$x(t) = x_T e^{-r(T-t)}.$$

## No-Arbitrage and Pricing

Arbitrage is the practice of making profit off of price inconsistencies. Examples can include

- buying a security in one market (e.g. the New York Stock Exchange) and selling it in another (e.g. the Toky Stock Exchange) where its price is higher
- using financial derivatives to synthesize an effective security, which is equivalent to another security, in terms of market mechanics
- finding "errors" in security pricing by a brokerage.

Arbitrage opportunities do exist, but they quickly disappear. As soon as someone spots an arbitrage opportunity, they exploit it until the price discrepancy goes away. For example, if a security is priced too low on exchange A and too high on exchange B, the arbitrageur buys on A and sells on B. This drives the A price up and the B price down, until the discrepancy is gone.

Therefore the following assumption is roughly valid.

*Assumption A.1 No arbitrage possibilities exist.*

This is a powerful principle in financial mathematics.

In addition we make the following simplifying assumptions:

**A.2** There are no transaction costs or taxes.

**A.3** All market participants can borrow and lend at the same risk-free rate of return $r$.

**A.4** There are no dividends or interest payments on the securities.

**A.5** The markets are perfectly elastic, i.e. a limitless number of shares of a security can be bought or sold at the prevailing price.

Each of these assumptions has its limitations, but for a start they are pretty good.

As a simple no-arbitrage argument, (Hull, Sect 3.2) we determine the value (i.e. the price) $f$ for a long forward contract. The forward contract is an agreement to buy the security at price $X$ and at time $T$.

Compare two portfolios at time $t$:

**A** One long forward contract, plus an amount of cash equal to $Xe^{-r(T-t)}$

**B** One unit of the security, whose price is denoted as $S(t)$.

I claim that these two portfolios are equivalent financially. The reason is that at time $T$, portfolio A will consist of the forward contract and an amount of cash. The cash will have grown to an amount

$$X = (Xe^{-r(T-t)})e^{r(T-t)}$$

over the time period $(T-t)$ at rate $r$. At this time the holder of the future contract is required to buy one share of the security for price $X$. So at time $T$ portfolio becomes one share, which is the same as portfolio B.

Now comes the no-arbitrage argument. If the value of portfolio A were less than the value of B, then buying A and shortselling B would generate cash today. At time $T$, the arbitrageur could sell A and buy B at no risk since both would consist of the single security unit. This would make money with no risk.

A similar procedure works if the price of A is higher than B.

Therefore price (A) = price (B), i.e. value (A) = value (B). Now

$$\begin{aligned} \text{value}(A) &= f(t) + Xe^{-r(T-t)} \\ \text{value}(B) &= S(t) \end{aligned}$$

So

$$f(t) = S(t) - Xe^{-r(T-t)}.$$

In particular, when a forward agreement is initiated, the strike price $X$, which is called $F$ is adjusted so that the value $f$ is 0, i.e.

$$0 = S - Fe^{-r(T-t)}$$

i.e.

$$F = Se^{r(T-t)}$$

is the forward price for an agreement at time $t$ to purchase a security, whose present price is $S$, at time $T$ for the price $F$.

## Properties of Stock Option Prices (Hall, Chapt. 7)

A *put option* give the holder the right (but not the obligation) to sell the stock at time $T$, the expiration time, for a specified price, the strike price $X$.

A *call option* gives the holder the right (but not the obligation) to buy the stock at time $T$ at price $X$.

These are called European options. American options allow exercise at any time up to the expiration time $T$.

Denote the value of European calls and puts by $c$ and $p$; and the values of American calls and puts by $C$ and $P$. The underlying security's price is $S$.

The value, or cashflow, at the exercise time is simple to work out. For a put option, the holder would prefer to sell the security if its price $S$ is smaller than the strike price $X$. Otherwise, the holder would be better off not exercising the option. In order to exercise the option (when $X > S$), the holder must first purchase it, paying an amount $S$. Thus in this case, the holder pays amount $S$ and receives amount $X$.

So the value of a put at exercise time is

$$p = \begin{cases} X - S & \text{if} \quad X > S \\ 0 & \text{if} \quad X < S \end{cases} \tag{1}$$

i.e.

$$p = \max(X - S, 0).$$

Similarly the value of a call at exercise is

$$c = \begin{cases} S - X & \text{if} \quad X < S \\ 0 & \text{if} \quad X > S \end{cases} \tag{2}$$

i.e.

$$c = \max(S - X, 0)$$

Now we derive some useful relationships between values of calls, puts and the underlying security. First, the call, which entitles the holder to buy as share of the equity, can never be worth more than the stock, i.e.

$$c \leq S.$$

Similarly, the put allows the holder to sell the shock for price $X$. So it can't be worth more than $X$, i.e.

$$p \leq X.$$

Lower bounds on $p$ and $c$ are obtained by comparing various portfolios. For a call, the argument goes as follows: Consider two portfolios

**Portfolio A:** One call and cash $Xe^{-r(T-t)}$

**Portfolio B:** One share.

At time $T$ of exercise, $A$ is worth

- $((S-X)$ from call$)$ + $(X$ from cash$)$ = $S$ if exercise occurs.
- (0 from call) + $(X$ from cash$)$ = $X$ if no exercise.

So at time $T$

$$\begin{aligned} \text{value}(A) &= \max(S, X) \\ &\geq \text{value}(B) = S \end{aligned}$$

By no-arbitrage, for all $t$

$$\text{value}(A) \geq \text{value}(B)$$

i.e.

$$c + Xe^{-r(T-t)} \geq S$$

so that

$$c \geq S - Xe^{-r(T-t)}.$$

Since also $c \geq 0$, then

$$c \geq \max(0, S - Xe^{-r(T-t)}).$$

A similar argument for puts yields the result

$$p \geq \max(0, Xe^{-r(T-t)} - S).$$

Next we derive the put-call parity result. Again, we use two portfolios.

**Portfolio A:** One call option, plus cash $Xe^{-r(T-t)}$

**Portfolio B:** One put option, plus one share.

Both of these have value $\max(s, x)$ at time $T$. The values of A and B at time $T$ are

$$\begin{aligned}
\text{value}(A) &= \text{value (call)} + X \\
&= \max(S - X, 0) + X \\
&= \max(S, X) \\
\text{value}(B) &= \text{value (put)} + S \\
&= \max(X - S, 0) + S \\
&= \max(X, S).
\end{aligned}$$

So at time $t = T$, value(A)=value(B). Then by the no-arbitrage assumption, at any time $t$

$$\text{value}(A) = \text{value}(B)$$

i.e.

$$c + Xe^{-r(T-t)} = p + S$$

This is the put-call parity formula.