Math 181 **Lecture 4**

Random Walks with Gaussian Steps

In this lecture we discuss random walks in which the steps are continuous normal (i.e. Gaussian) random variables, rather than discrete random variables. While this loses the simplicity of the random walk on a lattice, as discussed in the previous lecture, it gains in uniformity; the distribution of values at each time step is always Gaussian. This will be a useful property in modeling stock prices, which will be the subject of Lecture 5.

Outline of Lecture 4

(4.1) Additive property of Gaussian random variable

(4.2) Random walks with Gaussian increments

(4.3) Comparison of random walks with discrete and with Gaussian steps.

4.1 Additive property of Gaussian Random Variable

The following property of Gaussian random variables makes them easy to use

Theorem 4.1

If x and y are independent Gaussian random variables with mean m_x and m_y and variance σ_x^2 and σ_y^2, respectively, then $z = x + y$ is also Gaussian with mean and variance

$$\begin{aligned} m_z &= m_x + m_y \\ \sigma_z^2 &= \sigma_x^2 + \sigma_y^2. \end{aligned}$$

Proof First we derive this from a direct calculation, then we show the reasons why it is true.

The formulas for m_z and σ_z are easy

$$\begin{aligned} m_z &= E(z) = E(x+y) = E(x) + E(y) = m_x + m_y, \\ \sigma_z^2 &= E((z-m_z)^2) \\ &= E(((x+y) - (m_x+m_y))^2) \end{aligned}$$

$$\begin{aligned}
&= E(((x-m_x)+(y-m_y))^2) \\
&= E((x-m_x)^2+(x-m_x)(y-m_y)+(y-m_y)^2) \\
&= \sigma_x^2+\sigma_y^2.
\end{aligned}$$

The middle time vanished because of independence, i.e. since x and y are independent

$$\begin{aligned}
E((x-m_x)(y-m_y)) &= E(x-m_x)E(y-m_y). \\
&= 0
\end{aligned}$$

The more difficult step is to show the Gaussian distribution for z. For simplicity set $m_x = m_y = 0$ and $\sigma_x = \sigma_y = 1$. Calculate

$$\begin{aligned}
P(a<z<b) &= P(a<x+y<b) \\
&= \int\int_{a<x+y<b} p_x(x)p_y(y)dxdy \\
&= \int_{-\infty}^{\infty}\int_{a-x}^{b-x}(2\pi)^{-1}e^{-x^2/2}e^{-y^2/2}dxdy.
\end{aligned}$$

Change variable from (x,y) to $(z=x+y, w=x-y)$ i.e. $x=(z+w)/2,\ y=(z-w)/2$. The determinant $\det(\partial(x,y)/\partial(w,z)=\frac{1}{2}$. For $a<x+y<b$, we get $a<z<b$ and $-\infty<w<\infty$ so that

$$\begin{aligned}
P(a<z<b) &= \int_a^b\int_{-\infty}^{\infty} p_x((z+w)/2)p_y((z-w)/2)\frac{1}{2}dwdz \\
&= \int_a^b\int_{-\infty}^{\infty}(2\pi)^{-1}e^{-(z+w)^2/8}e^{-(z-w)^2/8}dwdz \\
&= \int_a^b\int_{-\infty}^{\infty}(2\pi)^{-1}\exp\{-\frac{1}{8}((z^2+2zw+w^2)+(z^2-2zw+w^2))\}\frac{1}{2}dwdz \\
&= \frac{1}{2}(2\pi)^{-1}\int_a^b\int_{-\infty}^{\infty}e^{-\frac{1}{4}z^2}e^{-\frac{1}{4}w^2}dwdz.
\end{aligned}$$

Now $\frac{1}{(4\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty}e^{-\frac{1}{4}w^2}dw = 1$, since the integrand is the density of a normal variable with $\sigma^2=2$. This gives

$$P(a<z<b) = (2\pi)^{-\frac{1}{2}}\int_a^b e^{-z^2/4}dz$$

which shows that z has a Gaussian density with $\sigma_z^2 = 2$.

Now an intuitive explanation of this calculation. As Gaussians, x and y can both be written as the limits of sums of IID random variable. For simplicity, we again set $m_x = m_y = 0$, $\sigma_x = \sigma_y = 1$. Then

$$\begin{aligned} x &= \lim_{N\to\infty} \{\frac{1}{\sqrt{N}} \sum_{n=1}^{N} a_n\} \\ y &= \lim_{N\to\infty} \{\frac{1}{\sqrt{N}} \sum_{n=1}^{N} b_n\} \end{aligned}$$

so that

$$z = \lim_{N\to\infty} \frac{1}{\sqrt{N}} \sum_{n=1}^{N} (a_n + b_n).$$

This shows that z is also a limit of sums of IID variables so that z must be Gaussian.

Finally note that a sum of any number of Gaussian is again Gaussian. To see this just add together two at a time, $z = (((x_1 + x_2) + x_3) + x_4)$ for example.

Random walks with Gaussian Steps

Consider the following random walk

$$\begin{aligned} x_0 &= 0 \\ x_n &= \sum_{i=1}^{n} d_i \end{aligned}$$

in which d_i are IID Gaussian with mean 0 and variance 1. From the prior discussion we know each x_n is Gaussian with mean 0 and variance $\sigma_x^2 = n$, i.e. $x_n = \sqrt{n}\nu$ in which ν is $N(1,0)$. As a function of n, the probability densities broaden as n increases without changing their shape.

Since $\sigma_{x_n} = \sqrt{n}$ is a measure of the deviation of x from its mean 0, then the size of x_n is $O(\sqrt{n})$. What's striking about this is that we've added together n terms d_i, each of which is $\mathcal{O}(1)$, but the result is not of size $\mathcal{O}(n)$ but rather $\mathcal{O}(\sqrt{n})$. This is due to cancelations of the d_i, since they come with both positive and negative values. This cancelation is at the heart of the Central Limit Theorem.

4.3. Comparison of a Random Walk with Discrete Steps to One with Gaussian Steps

Let x_n be a random walk with Gaussian step d_i; y_n a random walk with discrete steps c_i. For n large, these are nearly the same. As shown in Lecture 3, the CLT implies that

$$y_n \sim \sqrt{n}\omega \quad \text{as} \quad n \to \infty$$

in which ω is $N(0,1)$. But also $x_n = \sqrt{n}\nu$ in which ν is $N(0,1)$. Therefore x_n and y_n have (almost) the same statistics when n is large.

In the sequel we prefer to use x_n because it is a Gaussian for all n; whereas y_n only becomes Gaussian as $n \to \infty$.

Random Walk Models for Stock Prices

5.1. A Model for Stock Prices

Consider the price S_n for a stock at time $t_n = ndt$. As discussed in Lecture 1, the times t_n could be daily or they could be on some other relevant time scale. As described before, we expect the value of the stock to grow exponentially, so that $\log S_n$ grows linearly in time. That's not all, however; the evolution of S_n also has a random component.

The resulting model for the evolution of stock price is

$$\log S_{n+1} = \log S_n + \alpha + \beta d_{n+1}$$

in which α and β are constants and d_n is a random variable. As in Lecture 1, we can repeatedly use this equation to obtain

$$\log S_n = \log S_0 + n\alpha + \beta \sum_{i=1}^{n} d_i$$

which can be exponentiated to get

$$\begin{aligned} S_n &= S_0 \exp\{n\alpha + \beta \sum_{i=1}^{n} d_i\} \\ &= S_0 \exp\{\gamma t_n + \beta \sum_{i=1}^{n} d_i\} \end{aligned}$$

in which $\alpha = \gamma dt$.

We consider two main possibilities

(i) $d_i = \pm 1$ with probability $\left(\frac{1}{2}, \frac{1}{2}\right)$. Then

$$S_n = S_0 e^{\gamma t_n + \beta y_n}$$

in which y_n is a discrete random walk.

(ii) $d_i = \nu_i$ which are IID $N(0, 1)$ variables. Then

$$S_n = S_0 e^{\gamma t_n + \beta x_n}$$

in which x_n is a random walk with Gaussian increments.

Now set

$$\begin{aligned} \gamma &= \mu - \sigma^2/2 \\ \beta &= \sigma\sqrt{dt}. \end{aligned}$$

For the Gaussian case, as shown in the last lecture

$$x_n = \sqrt{n}\omega_n$$

in which ω_n is $N(0,1)$, so that

$$\begin{aligned} \beta x_n &= \sigma\sqrt{ndt}\omega_n \\ &= \sigma\sqrt{t_n}\omega_n. \end{aligned} \tag{1}$$

Then

$$\log S_n = \log S_0 + ((\mu - \sigma^2/2)t_n + \sigma\sqrt{t_n}\omega_n). \tag{2}$$

and

$$S_n = S_0 \exp((\mu - \sigma^2/2)t_n + \sigma\sqrt{t_n}\omega_n) \tag{3}$$

In this formula for S_n, μ is the average growth rate in time. In particular

$$\begin{aligned} E(S_n) &= S_0 \exp((\mu - \sigma^2/2)t_n)\ E[\exp(\sigma\sqrt{t_n}\omega_n)] \\ &= S_0 \exp((\mu - \sigma^2/2)t_n)\ \exp((\sigma^2/2)t_n) \\ &= S_0 \exp(\mu t_n). \end{aligned}$$

This looks just like the formula for growth of a portfolio due to compound interest, except that the interest rate r is replaced by the average growth rate μ.

The constant σ, which is called the volatility, measures the amount of *risk* in the dynamics of S_n. In particular

$$\begin{aligned} \mathrm{var}(\log S_n) &= E((\log S_n - E(\log S_n))^2) \\ &= E(\sigma\sqrt{t_n}\omega_n)^2) \\ &= \sigma^2 t_n E(\omega_n^2) \\ &= \sigma^2 t_n. \end{aligned}$$

This shows that the variance in $\log S_n$ grows linearly in time with coefficient σ^2.

5.2. The Price of Risk

If $\sigma = 0$, then the asset value grows without any risk. Although that's not a good model for a stock, it is correct for a U.S. Treasury bond or an insured bank deposit. These are *risk-free* assets, with a risk-free rate of return $\mu = r$.

The price evolution of any other asset can be compared to that of a risk-free asset. For an asset with volatility σ, the difference $\mu - r$ is called the *risk premium* and the ratio $(\mu - r)/\sigma$ is called the *price of risk*. It is the extra average growth rate that is required to get investors to buy the risky asset with volatility σ, per unit of volatility, instead of investing at the risk-free rate.

With some reflections, you might ask whether the price of risk $(\mu - r)/\sigma$ is always positive. The answer is no. Certainly gambling games in casinos or state lotteries have a negative expected return, i.e. $\mu < 0$, and a positive amount of risk $\sigma > 0$. So they have a negative price of risk. This means that a gambler pays for the possibility of winning a lot.

On the other hand, the stock market rewards investors, at least in theory, so that $\mu > r$. This is the difference between gambling and investing.

Deficiencies in the Model

The main deficiencies in this model have to do with large changes in the value of S_n. This model does not correctly account for crashes or even for large changes (say of size 5%) in stock prices.