Lecture 22 Models of the short rate

Equilibrium models

(1) log normal model $\quad dr_n = \mu r_n\, dt + \sigma r_n\, \omega_n \sqrt{dt}$

$$r_n \cong r_{n-1}\, e^{(\mu - \sigma^2/2)\, dt + \sigma \sqrt{dt}\, \omega_n}$$

In this model, interest rates have no long term memory. They can get large or small.

(2) Vasicek model – mean reversion to a historical value

$$dr_n = a(b - r_n)\, dt + \sigma\, \omega_n \sqrt{dt}$$

In this model, the short term rate r, tends to return to the value b. This mean reversion is observed in markets. This model has the defect that r_n can become negative which is nonsensical

(3) Cox-Ingersoll-Ross model

$$dr_n = a(b - r_n)\, dt + \sigma \sqrt{r_n}\, \omega_n \sqrt{dt}$$

Now the solution cannot become negative, as r_n approaches 0, the amplitude of the random terms becomes too small to make the solution negative, at least in the $dt \to 0$ limit.

Obtaining term structure from short rate. Consider an interval $[t, T]$. Let $\bar{r}$ = average of $r = \frac{1}{T-t}\int_t^T r(t')\,dt'$

The t price of bond paying off \$1 at time T is

$$P(t,T) = \bar{E}\left[e^{-\bar{r}(T-t)}\right].$$

$$= e^{-R(t,T)(T-t)}$$

Thus the term structure is given by

$$R(t,T) = -\frac{1}{T-t}\log \bar{E}\left[e^{-\bar{r}(T-t)}\right].$$

Two factor models. These models describe dynamics for both the short rate r and a long term rate L. The short term rate tends to the long term rate.

$$dr_n = (a_1 + b_1(L_n - r_n))\,dt + r_n \sigma_1 \omega_n \sqrt{dt}$$

$$dL_n = L_n(a_2 + b_2 r_n + c_2 L_n)\,dt + L_n \sigma_2 \nu_n \sqrt{dt}$$

The next step with these models is to calibrate them to the market and to predict the resulting term structure and its evolution. The results are good but limited. There are a small number of parameters allowing term structure curves of reasonable flexibility. The dynamics of the term structure does not match well the observed dynamics in the market.

Lecture 23 HJM and LMM models

In order to overcome the deficiencies of short rate models the most recent models allow more flexibility in matching the term structure.

The first of these models is the Heath-Jarrow-Morton (HJM) model. It directly describes the price $P(t,T)$ at t for a bond paying \$1 at time T. With T fixed

$$dP(t,T) = r(t)P(t,T)dt + v(t,T)P(t,T)\,dz(t)$$

in which $dz = \omega\sqrt{dt}$. Here $r(t)$ is the short rate and $v(t,T)$ is the volatility for $P(t,T)$. From $P(t,T)$, one can find the forward rate $F(t,T_1,T_2)$

$$f(t,T_1,T_2) = \frac{\log P(t,T_1) - \log P(t,T_2)}{T_2 - T_1}$$

and the instantaneous forward rate $F(t,T)$

$$F(t,T) = \frac{\partial}{\partial T}\log P(t,T)$$

It satisfies an equation of the form

$$dF(t,T) = m(t,T)dt + s(t,T)\,dz(t)$$

with

$$m(t,T) = v(t,T)\,v_T(t,T)$$

$$s(t,T) = -v_T(t,T)$$

so that

$$m(t,T) = s(t,T)\int_t^T s(t,\tau)\,d\tau$$

The LIBOR Market Model or LMM was initiated by Brace, Gatarek and Musiela and by Miltersen, Sandmann and Sondermann. It models the forward rate $F_k(t)$ over a period (t_k, t_{k+1}) in a way that can be directly related to the observed prices of bonds (through the LIBOR = London Inter-Bank Offer Rate).