Lecture 21 Black's Model for Option Prices

Consider a call option on a security with value V.

~~Consider~~ Define

T = time to maturity

F = forward price of V for contract with maturity T

F_0 = value of F at $t=0$

K = strike price of the option

$P(t,T)$ = price at time t of (zero-coupon) bond paying \$1 at T

V_T = value of V at T

σ = volatility of F

Assume

- $\log V_T$ is normal with mean F_0 and variance $\sigma^2 T$
- the discount factor over the period $[0,T]$ is $P(0,T)$

The payout of the option at time T is $\max(V_T - K, 0)$. Since V_T is lognormal,

$$E[\max(V_T - K, 0)] = E[V_T]N(d_1) - KN(d_2)$$

$$d_1 = \frac{\log(E[V_T]/K) + \sigma^2 T/2}{\sigma\sqrt{T}} = \frac{\log(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\log(E[V_T]/K) - \sigma^2 T/2}{\sigma\sqrt{T}} = \frac{\log(F_0/K) - \sigma^2 T/2}{\sigma\sqrt{T}}$$

as in call price. Since $E[V_T] = F_0$ and discounting is by $P(0,T)$ then

$$c = P(0,T)\{F_0 N(d_1) - KN(d_2)\}$$

The solution comes from evaluating

$$\text{Prob}(V_T > k) = N(d_2)$$

$$E[V_T \mid V_T > k] = E[V_T]\,N(d_1)$$

Application to value of cap (or caplet, since it is for a single payment) with payoff

$$\max(R - R_K, 0)$$

R_K = strike

R_T = interest rate

= log normal volatility σ

F_T = forward rate

$$\text{value} = P(0, T)\,[F_T\,N(d_1) - R_K\,N(d_2)]$$

$$d_1 = \frac{\log(F_T / R_K) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\log(F_T / R_K) - \sigma^2 T/2}{\sigma\sqrt{T}}$$