Lecture 19

Derivative Securities Depending on Multiple Underlyings

We now formulate risk-neutral pricing for an option that depends on multiple underlying securities. An example is a basket option. Consider M securities $S^{(1)}, \ldots, S^{(M)}$ and form the average

$$\bar{S} = \frac{1}{M} \sum_{k=1}^{M} S^{(k)}$$

Then a call on a basket option has payout

$$c_{\text{basket}}(T) = \max(0, \bar{S}(T) - X).$$

In order to analyze these, we first need a model for multiple securities. In general the changes in value for different stocks will be correlated. So the normal increments in the different random walks must be correlated.

We start by defining a multi-variable normal distribution

$$\omega = \begin{pmatrix} \omega^{(1)} \\ \cdots \\ \omega^{(M)} \end{pmatrix}.$$

Define the covariance matrix Σ as an $M \times M$ matrix with

$$\Sigma_{ij} = E[\omega^{(i)} \omega^{(j)}]$$

which can be written in matrix-vector rotation as

$$\Sigma = E[\omega \omega^{+}]$$

in which $\omega^{+} = (\omega^{(1)}, \ldots, \omega^{(M)})$ the transpose of ω.

Form the following quadratic sum

$$\omega^{+} \Sigma^{-1} \omega = \Sigma_{ij} \omega^{(i)} (\Sigma^{-1})_{ij} \omega^{(j)}$$

in which Σ^{-1} is the inverse matrix to Σ. Then the multi-variable density function for ω is

$$p(\omega) = (2\pi)^{-1/2} |\det(\Sigma)|^{-1/2} e^{-\omega^{+} \ \Sigma^{-1} \omega / 2}.$$

In practice, the multi-variable Gaussian vector ω is formed as follows: Let ν be a vector of independent $N(0,1)$ random variable

$$\nu = \begin{pmatrix} \nu^{(1)} \\ \cdots \\ \nu^{(M)} \end{pmatrix}.$$

i.e.

$$\begin{aligned} &E(\nu_i^2) = 1 \\ &E(\nu_i \nu_j) = 0 \quad \text{if} \quad i \neq j \end{aligned}$$

i.e.

$$E(\nu\nu^+) = I.$$

Look for $\omega = A\nu$. Then we find that

$$\begin{aligned} \Sigma &= E(\omega\omega^+) \\ &= E(A\nu\nu^+ A^+) \\ &= AE(\nu\nu^+)A^+ \\ &= AIA^+ \\ &= AA^+. \end{aligned}$$

This shows that

$$\omega = A\nu$$

if

$$\Sigma = AA^+.$$

For a given matrix Σ, there are various numerical methods for finding such a factorization A.

Now we formulate a model for multiple securities. Let $\omega_i = (\omega_i^{(1)}, \ldots, \omega_i^{(M)})$ be independent samples from the multi-variable Gaussian distribution. Equivalently

$$\begin{aligned} \omega_i &= A\nu_i \\ \nu_i &= (\nu_i^{(1)}, \ldots, \nu_i^{(M)}) \end{aligned}$$

in which $\nu_i^{(k)}$ are independent $N(0,1)$ random variables. The exponential random walk model is

$$S_n^{(k)} = S_0^{(k)} \exp\{(u^{(k)} - {\sigma^{(k)}}^2/2)t_n + \sqrt{dt}\sum_{i=1}^{n}\omega_i^{(k)}\}$$

in which

$$\begin{aligned} \mu^{(k)} &= \text{average growth rate for } \; S^{(k)} \\ {\sigma^{(k)}}^2 &= \Sigma_{kk} = E(\omega^{(k)}\omega^{(k)}). \end{aligned}$$

Using the ν representation, we can write this as

$$S_n^{(k)} = S_0^{(k)} \exp\{(\mu^{(k)} - {\sigma^{(k)}}^2/2)t_n + \sqrt{dt}\sum_{i=1}^{n}\sum_{\ell=1}^{M} A_{k\ell}\nu_i^{(\ell)}\}.$$

The risk neutral process if formed by replacing each of the $\mu^{(k)}$'s by r, i.e.

$$S_n^{(k)} = S_0^{(k)} \exp\{(r - {\sigma^{(k)}}^2/2)t_n + \sqrt{dt}\sum_{i=1}^{n}\sum_{\ell=1}^{M} A_{k\ell}\nu_i^{(\ell)}\}$$