Math 181 Lecture 18

American Options (Hull 7.4, 7.5)

American options can be exercised at any time up to and including the expiration date. For an American call or put, the decision to exercise or hold at any time t depends just on the time value t and the underlying stock value $S(t)$. The exercise time τ is chosen to maximize the value of the option.

For an American call (on a stock without dividends), early exercise is never optimal. The reason is that exercise requires payment of the strike price X. By holding onto X until the expiration time, the option holder saves the interest on X.

To see this mathematically, consider two portfolios

$$\begin{array}{ll} E: & \text{one American call } c, Xe^{-r(T-t)} \text{ cash} \\ F: & \text{one share } S. \end{array}$$

If the exercise time is $\tau < T$, the value of E is

$$\begin{aligned} E &= (S - X) + Xe^{-r(T-\tau)} \\ &< S = F. \end{aligned}$$

If the exercise is at $\tau = T$, then

$$\begin{aligned} E &= \max(S - X, 0) + X \\ &= \max(S, X) \\ &\geq S = F. \end{aligned}$$

It follows that $E \geq F$ for all times, so that one should never take $\tau < T$.

For an American put (or an American call on a stock with dividends) early exercise is sometimes optimal. Suppose for example, that the stock price S falls to nearly 0. Then the option holder stands to gain more by exercise than by waiting. The reason is that the payout $X - S$ cannot increase much, but by early exercise, the option holder will get the interest on the payout.

Since the early exercise decision for an America depends only on t and $S(t)$, there is an early exercise boundary $S = B(t)$. For $S(t) > B(t)$, it is better to hold on to the option. If the stock price hits $S(t) = B(t)$, then the

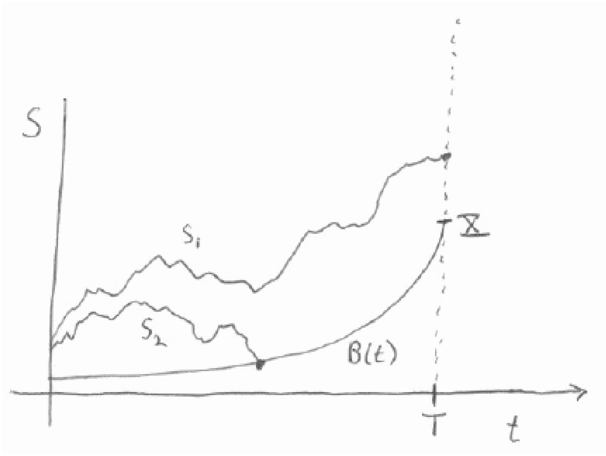

Figure 1: Early exercise boundary $S = B(T)$ and two paths S_1 (with early exercise) and S_2 (without early exercise) for an American option.

put option should be exercised. (If $S(0) < B(0)$, then the option should be exercised when first issued).

Figure 1 shows the early exercise boundary for an American put $S = B(t)$. Two typical paths are also drawn. The first (S_1) will be exercised at the time that it hits the early exercise boundary; the second (S_2) never hits the early exercise boundary so that it is not exercised.

Another interesting feature, is that the slope $B'(T)$ of the boundary is infinite at the expiration time. It's local behavior is given by

$$B(t) = c\sqrt{(T-t)\log(T-t)}.$$

The value F of an American option can be written using risk-neutral valuation in several ways. If the option is exercised at time t and stock price S, denote the payout function as $G(t,S)$. First the value F can be written as the (discounted) expectation of the payout at the time t_B that the underlying price S crosses the early exercise boundary B. For a given value of t, define

$$t_B = \min\{t' : t' > t \text{ and } S(t') = B(t')\}. \tag{1}$$

Then

$$F(t) = \bar{E}[e^{-r(t_B - t)} G(t_B, S(t_B))]. \tag{2}$$

An alternative expression is based on the time τ of exercise. At time t, the decision of whether to exercise the option or to defer exercise until later, i.e., the choice of $\tau = t$ or $\tau > t$, can depend only on knowledge of the stock price $S(t')$ for $t' \leq t$. This property is call *causality* because it says that an action such as early exercise cannot depend on knowledge of future random events. A random time variable τ satisfying causality is called a *stopping time.*

Using the concept of a stopping time, the value F of an American option can be written using risk-neutral valuation. The value is the (discounted) expectation of the payout G at the time τ of exercise, in which the time τ is a stopping time chosen to maximize the value F. In other words

$$F(t, S_t) = \max_{\tau} \bar{E}[e^{-r(\tau-t)} G(\tau, S(\tau))] \tag{3}$$

in which the maximum is taken over all stopping times with $\tau \geq t$ and the underlying price is starts at time t as $S(t) = S_t$.

The expression (3) also provides a way of describing the decision of whether or not to exercise at time t and stock price S_t. If the option is exercised then the resulting value is $G(t, S_t)$. On the other hand, if exercise is deferred until later then the value is the right hand side of (3). Thus early exercise should be performed if

$$G(t, S_t) \geq \max_{\tau \geq t} \bar{E}[e^{-r(\tau-t)} G(\tau, S(\tau))]. \tag{4}$$

In addition, we express the value F at any point (t, S_t) as

$$F(t, S_t) = \max \{G(t, S_t), H(t, S_t)\} \tag{5}$$

in which

$$H(t, S_t) = \bar{E}[\text{discounted value of } F \text{ if exercise is deferred}]. \tag{6}$$

The formula (5) provides a direct method for evaluating an American option on a tree (i.e., a binary random walk). Suppose the tree consists of n time steps, with time values $t_k = kdt$. Start at the final time $t_n = T$. Since there is no possibility of deferral to a later time, then the value of the American option is just the payout; i.e., $F(t_n, S_n) = G(t_n, S_n)$. Now solve backwards, at time t_{n-1} we can directly evaluate G_{n-1} and $H_{n-1} = H(t_{n-1}, S_{n-1})$ as

$$\begin{aligned} G_{n-1} &= G(t_{n-1}, S_{n-1}) && (7) \\ H_{n-1} &= \bar{E}[e^{-rdt} F_n] = e^{-rdt} \left(pF(t_n, uS_{n-1}) + (1-p)F(t_n, dS_{n-1}) \right). && (8) \end{aligned}$$

The value of F_{n-1} is then

$$F_{n-1} = \max \left\{ G_{n-1}, H_{n-1} \right\}. \qquad (9)$$

For any k, the formulas (7), (8) and (9), can be used to obtain the values of F at $k-1$, once the values of F are known at k. In this way, all of the values of F can be found. This is called “rolling back along the tree.”