Math 181 Lecture 14

Risk-Neutral Valuation

The most surprising feature of the Black-Scholes equation, or of the Black-Scholes formulas, is that they do not involve the average rate of return μ. No matter what value μ takes, the value of the option is the same for given values of σ and r. This has led to a method called "risk-neutral valuation" for finding option prices. We explain this first in a descriptive way and then in a mathematical way.

In the real world $\mu > r$ if $\sigma > 0$; since investors are risk-averse, they require a greater rate of return to assume the volatility σ. As discussed in the binomial tree sections, this is the reason that the option price is not the expected value

$$f(S,t) \neq e^{-r(T-t)} E(f(S_T, T)). \tag{1}$$

We may imagine a world in which all investors are risk-neutral. That is, in which $\mu = r$ for any σ. In the imaginary "risk-neutral world", the option price is the average, i.e.

$$f(S,t) = e^{-r(T-t)} \bar{E}(f(S_T, T)) \tag{2}$$

in which $\bar{E}$ is the average in the risk-neutral world. Moreover, Black-Scholes showed that the option price is the same in the real world and in the risk-neutral world. So (2) gives the correct price in the real world, using an average in the risk-neutral world. This pricing method is referred to as "risk-neutral valuation".

In order to make the average in (2) explicit, we need a probabilistic representation of $S(t)$. The Gaussian random walk for S is

$$dS_{n+1} = \mu S_n dt + \sigma S_n \sqrt{dt} \nu_{n+1}.$$

Now consider $\log S_{n+1} = y_{n+1} = Y(S_{n+1})$ with

$$Y(S) \;=\; \log S,\; Y_s = S^{-1},\; Y_{ss} = -S^{-2}.$$

Now expand

$$
\begin{aligned}
dy_{n+1} &= Y(S_{n+1}) - Y(S_n) \\
&\cong Y_s dS_{n+1} + \frac{1}{2} Y_{ss} (dS_{n+1})^2 \\
&= S_n^{-1} dS_{n+1} - \frac{1}{2} S_n^{-2} (dS_{n+1})^2 \\
&= S_n^{-1} (\mu S_n dt + \sigma S_n \sqrt{dt}\ \nu_{n+1}) \\
&\quad - \frac{1}{2} S_n^{-2} (\mu S_n dt + \sigma S_n \sqrt{dt}\ \nu_{n+1})^2 \\
&= \mu dt + \sigma \sqrt{dt} \nu_{n+1} - \frac{1}{2} (\mu dt + \sigma \sqrt{dt}\ \nu_{n+1})^2 \\
&= \mu dt + \sigma \sqrt{dt}\ \nu_{n+1} - \frac{1}{2} \sigma^2 \nu_{n+1}^2 dt + \mathcal{O}(dt^{3/2}) \\
&\cong \mu dt + \sigma \sqrt{dt}\ \nu_{n+1} - \frac{1}{2} \sigma^2 dt
\end{aligned}
$$

as in the Black-Scholes derivation. Therefore

$$
dy_{n+1} = \tilde{\mu} dt + \sigma \sqrt{dt}\ \nu_{n+1}
$$

in which

$$
\tilde{\mu} = \mu - \frac{1}{2} \sigma^2. \tag{3}
$$

Add this up to get

$$
\begin{aligned}
y_{n+1} &= (y_{n+1} - y_n) + (y_n - y_{n-1}) + \cdots + (y_1 - y_0) + y_0 \\
&= y_0 + \sum_{k=1}^{n+1} dy_k \\
&= y_0 + \tilde{\mu}(n+1) dt + \sigma \sqrt{dt} \sum_{k=1}^{n+1} \nu_{n+1} \\
&= y_0 + \tilde{\mu} t_{n+1} + \sigma \sqrt{t_{n+1}}\ \omega
\end{aligned}
$$

in which ω is an $N(0,1)$ random variable. This used the additive property of Gaussian random variables.

This says that

$$
\log S(t_{n+1}) = \log S(0) + \tilde{\mu} t_{n+1} + \sigma \sqrt{t_{n+1}} \omega.
$$

Apply exp to obtain

$$
S(t_{n+1}) = S(0) e^{\tilde{\mu} t_{n+1} + \sigma \sqrt{t_{n+1}}\ \omega}.
$$

Replace t_{n+1} by t and $\tilde{\mu}$ by $\mu - \sigma^2/2$

$$S(t) = S(0)e^{(\mu-\sigma^2/2)t+\sigma\sqrt{t}\ \omega}. \tag{4}$$

This formula for $S(t)$ gives a probabilistic representation, since ω is an $N(0,1)$ random variable. A slight extension is

$$S(T) = S(t)e^{(\mu-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}\ \omega} \tag{5}$$

for any $T > t$.

The formulas (4), (5) are for the real process, since they involve the growth rate μ, including the risk premium. In the risk-neutral world, the only difference is that μ is replaced by r. This says that the risk neutral representation for $S(t)$ given $S(\bar{t})$ is

$$S(T) = S(t)e^{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}\ \omega}. \tag{6}$$

In the risk neutral valuation formula (2), this risk-neutral stock price is used, i.e.

$$f(S,t) = e^{-r(T-t)}E_\omega[f]$$

in which

$$f = f(S\exp\{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}\ \omega\}).$$