Math 181 Lecture 11

Options for Random Walks with Gaussian Increments

In this lecture we start on the valuation of options for a Gaussian random walk. As in the discrete random walk, we will construct a portfolio that is risk-free, or approximately risk free.

For the discrete walk, the risk-free portfolio was constructed only for a single step. Here we need to make it risk-free over many steps. For this to be true the hedging parameter needs to be changed as time progresses, which is called *dynamic hedging*.

We consider a *self-financing portfolio* A where value at time n is denoted A_n, and which consists of α_n units of stock with price S_n and β_n units of an option with price f_n. Thus

$$A_n = \alpha_n S_n + \beta_n f_n. \tag{1}$$

The evolution of A_n is in two steps. First, the values of S_n and f_n change to (S_{n+1}, f_{n+1}). Second, the portfolio is rebalanced, i.e. α_n and β_n are changed to $(\alpha_{n+1}, \beta_{n+1})$. This rebalancing is chosen to be self-financing, i.e. it does not require (or produce) cash. This says that

$$\alpha_n S_{n+1} + \beta_n f_{n+1} = \alpha_{n+1} S_{n+1} + \beta_{n+1} f_{n+1} = A_{n+1}. \tag{2}$$

The left hand side is the value after the change in stock and option values, but before the rebalancing. The right hand side is after rebalancing.

It follows that the change in portfolio value is

$$\begin{aligned} dA_{n+1} &= A_{n+1} - A_n \\ &= (\alpha_n S_{n+1} + \beta_n f_{n+1}) - (\alpha_n S_n + \beta_n f_n) \\ &= \alpha_n dS_{n+1} + \beta_n df_{n+1} \end{aligned} \tag{3}$$

in which

$$\begin{aligned} dS_{n+1} &= S_{n+1} - S_n \\ df_{n+1} &= f_{n+1} - f_n. \end{aligned}$$

At this point, we are ready to invoke the Gaussian random walk model for the stock price. This can be phrased in terms of S_n or $\log S_n$ (an important distinction between these two will be explained later).

Here we use S_n. The model for stock price evolution is

$$S_{n+1} = (1 + \mu dt)S_n + \sigma\sqrt{dt}\omega_{n+1}S_n$$

in which the ω_n are IID, $N(0,1)$ random variables. Therefore

$$dS_{n+1} = \mu S_n dt + \sigma S_n \sqrt{dt}\omega_{n+1}. \tag{4}$$

Next we assume that the option price f is a smooth function of stock price S and time t, i.e.

$$f = f(S,t).$$

Then

$$\begin{aligned} f_n &= f(S_n, t_n) \\ df_{n+1} &= f_{n+1} - f_n \\ &= f(S_{n+1}, t_{n+1}) - f(S_n, t_n). \end{aligned}$$

We will perform a Taylor series expansion for df_{n+1}. Determination of the significant terms in the expansion turns out to be surprisingly subtle, involving what is called *Ito's lemma* in probability theory.

Applying Taylor's expansion yields

$$\begin{aligned} df_{n+1} &= f_t(t_{n+1} - t_n) + f_s(S_{n+1} - S_n) \\ &\quad + \frac{1}{2} f_{ss}(S_{n+1} - S_n)^2 + \cdots \\ &= f_t dt + f_s dS_{n+1} + \frac{1}{2} f_{ss} dS_{n+1}^2 \\ &\quad + \mathcal{O}(dt^2 + dtdS + dS^3) \end{aligned}$$

in which f_t, f_s and f_{ss} are evaluated at (S_n, t_n).

Now according to the previous formula for dS_{n+1},

$$dS_{n+1} = \mu S_n dt + \sigma S_n \sqrt{dt}\omega_{n+1}.$$

In particular

$$\begin{aligned} dS &= \mathcal{O}(\sqrt{dt}) \\ \mathcal{O}(dt^2 &+ dtdS + dS^3) = \mathcal{O}(dt^{\frac{3}{2}}) \\ dS_{n+1}^2 &= (\mu S_n dt + \sigma S_n \sqrt{dt}\omega_{n+1})^2 \\ &= \sigma^2 S_n^2 dt \omega_{n+1}^2 + \mathcal{O}(dt^{\frac{3}{2}}). \end{aligned}$$

Combine all of this together to obtain

$$\begin{aligned} df_{n+1} &= f_t dt + f_s(\mu S_n dt + \sigma S_n \sqrt{dt}\omega_{n+1}) \\ &\quad + \frac{1}{2} f_{ss}\sigma^2 S_n^2 dt \omega_{n+1}^2 + \mathcal{O}(dt^{\frac{3}{2}}) \\ &= \sigma S_n f_s \omega_{n+1}\sqrt{dt} \\ &\quad + (f_t + \mu S_n f_s + \frac{1}{2}\sigma^2 S_n^2 f_{ss}\omega_{n+1}^2)dt \\ &\quad + \mathcal{O}(dt^{\frac{3}{2}}). \end{aligned} \tag{5}$$

The dominant, random term is the first one which is of size $\sqrt{dt}$. We choose α_n and β_n to eliminate this term. The change in value of the portfolio is (from (2))

$$\begin{aligned} dA_{n+1} &= A_{n+1} - A_n \\ &= \alpha_n dS_{n+1} + \beta_n df_{n+1} \\ &= \alpha_n \{\mu S_n dt + \sigma S_n \sqrt{dt}\omega_{n+1}\} \\ &\quad + \beta_n \{\sigma S_n f_s \omega_{n+1}\sqrt{dt} + (f_t + \mu S_n f_s + \frac{1}{2}\sigma^2 S_n^2 f_{ss}\omega_{n+1}^2)dt\} \\ &\quad + \mathcal{O}(dt^{\frac{3}{2}}) \\ &= (\alpha_n + \beta_n f_s)\sigma S_n \omega_{n+1}\sqrt{dt} \\ &\quad + (\beta_n f_t + (\alpha_n + \beta_n f_s)\mu S_n + \beta_n \frac{1}{2}\sigma^2 S_n^2 f_{ss}\omega_{n+1}^2)dt + \mathcal{O}(dt^{\frac{3}{2}}). \end{aligned}$$

Choose α_n, β_n to satisfy

$$\alpha_n + \beta_n f_s = 0$$

to eliminate the $\sqrt{dt}$ term. This also eliminates one of the dt terms.

The resulting equation for A is

$$dA_{n+1} = \beta_n(f_t + \frac{1}{2}\sigma^2 S_n^2 f_{ss}\omega_{n+1}^2)dt + \mathcal{O}(dt^{\frac{3}{2}}). \tag{6}$$

We also summarize the equations for α_n and β_n as

$$\alpha_n + \beta_n f_s = 0 \tag{7}$$

$$(\alpha_{n+1} - \alpha_n)S_{n+1} + (\beta_{n+1} - \beta_n)f_{n+1} = 0. \tag{8}$$

The second equation expresses the self-financing condition; the first is the condition that eliminates the dominant risk (i.e. randomness) in portfolio A.

Math 181 Lecture 12

Option Pricing and Ito's Lemma

In the previous lecture, we constructed a self-financing portfolio A, in which the dominant risk term $\sqrt{dt}\omega_n$ was eliminated. The resulting equation for the evolution of portfolio value is

$$\begin{aligned} dA_{n+1} &= A_{n+1} - A_n \\ &= \beta_n(f_t + \frac{1}{2}\sigma^2 S_n^2 f_{ss}\omega_{n+1}^2)dt + \mathcal{O}(dt^{\frac{3}{2}}). \end{aligned} \tag{9}$$

There is still randomness in the ω_{n+1}^2 term. In this lecture we show that the randomness in this turn is cancelled, using the Central Limit Theorem.

First we write ω_{n+1}^2 as a deterministic part, plus a mean 0 part. Since ω_{n+1} is $N(0,1)$ it has variance 1, this implies that

$$E(\omega_{n+1}^2) = 1$$

so that

$$\omega_{n+1}^2 = 1 + \tilde{\gamma}_{n+1} \tag{10}$$

in which

$$E(\tilde{\gamma}_{n+1}) = 0.$$

Insert (10) into (9) and denote

$$\begin{aligned} \gamma_{n+1} &= \beta_n \frac{1}{2}\sigma^2 S_n^2 f_{ss}\tilde{\gamma}_{n+1} \\ \kappa_{n+1} &= \beta_n(f_t + \frac{1}{2}\sigma^2 S_n^2 f_{ss}) \end{aligned}$$

to get

$$dA_{n+1} = \kappa_{n+1}dt + \gamma_{n+1}dt + \mathcal{O}(dt^{\frac{3}{2}}).$$

Now we use this to find A_n

$$\begin{aligned} A_n &= (A_n - A_{n-1}) + (A_{n-1} - A_{n-2}) + \cdots + (A_1 - A_0) + A_0 \\ &= A_0 + \sum_{k=1}^{n} dA_k \\ &= A_0 + \sum_{k=1}^{n} \kappa_k dt + \sum_{k=1}^{n} \gamma_k dt + \mathcal{O}(ndt^{\frac{3}{2}}). \end{aligned}$$

Now we analyze each term separately.

(i) The first sum contain n terms each of size dt, so it is size $\mathcal{O}(ndt) = \mathcal{O}(t_n)$.

(ii) The last term is $\mathcal{O}(ndt^{\frac{3}{2}}) = t_n\mathcal{O}(dt^{\frac{1}{2}})$. For fixed t_n, this term is close to 0 if dt is small.

(iii) The second sum, which is the most interesting term, looks at first just like the first sum. The difference, however is that each γ_k is random with

$$E(\gamma_k) = 0.$$

At this point we wish to apply the Central Limit Theorem to the sum of the γ_k. One complication is that $\tilde{\gamma}_k$ are IID, but γ_k are only approximately IID. Under this approximation, the Central Limit Theorem then says that

$$\sum_{k=1}^{n} \gamma_k = \mathcal{O}(\sqrt{n}).$$

So that

$$\begin{aligned} dt\sum_{k=1}^{n} \gamma_k &= \mathcal{O}(\sqrt{n}dt) \\ &= \mathcal{O}(\sqrt{ndt}\sqrt{dt}) \\ &= \sqrt{t_n}\mathcal{O}(\sqrt{dt}). \end{aligned}$$

As for the last term, this term is insignificant if t_n is fixed and dt is small.

To summarize, we have found that the γ_k and the $\mathcal{O}(dt^{\frac{3}{2}})$ terms are insignificant in the sum for A_n, in the limit that t_n is fixed and dt is small.

Therefore we drop these terms in the equation for dA_{n+1}; i.e.

$$\begin{aligned} dA_{n+1} &\simeq \kappa_{n+1}dt \\ &= \beta_n(f_t + \frac{1}{2}\sigma^2 S_n^2 f_{ss})dt. \end{aligned} \tag{11}$$

Finally, we have an equation for the value of portfolio A, with no randomness. This shows that, within this approximation (i.e. dt small) that A is a riskfree portfolio!

As a consequence, the no-arbitrage theory says that the value of A increases at the risk free rate, i.e.

$$dA_{n+1} = rA_n dt. \tag{12}$$

Now we can evaluate A_n using the formula (7) as

$$\begin{aligned} A_n &= \alpha_n S_n + \beta_n f_n \\ &= -\beta_n f_s S_n + \beta_n f_n \\ &= \beta_n (f - S_n f_s) \end{aligned}$$

so that

$$dA_{n+1} = \beta_n r (f - S_n f_s) dt. \quad (13)$$

Insert this into (11) to obtain

$$\beta_n r (f - S_n f_s) dt = \beta_n (f_t + \frac{1}{2}\sigma^2 S_n^2 f_{ss}) dt.$$

Cancel the β_n and dt factors and rearrange to obtain

$$-f_t = \frac{1}{2}\sigma^2 S^2 f_{ss} + r S f_s - r f. \quad (14)$$

This is the Black-Scholes equation, a partial differential equation for the price f of a derivative security.