Final Exam Solutions

3/17/03

1. Calculate

$$d_1 = \frac{1}{.2}\left\{ \log\left(\frac{100}{90}\right) + \left(.05 + .04/2\right) \right\}$$
$$= 5 \times \{.105 + .07\} = 5 \times \{.175\} = .875$$
$$d_2 = d_1 - .2 = .675$$

$$N(d_1) = N(.875) = .809$$
$$N(d_2) = N(.675) = .750$$

(a) $c = 100\, N(d_1) - (90/1.05)\, N(d_2)$
$= 16.6$

(b) $\Delta = N(d_1) = .809$

2. (a) Consider portfolio π consisting of forward agreement plus cash $Ke^{-r(T-t)}$. So that

$$\pi(t) = F(t) + Ke^{-r(T-t)}$$

At time $t = T$, this is $F(t) + K$. The forward agreement takes away the cash K and give one share of stock $S(T)$. Since $\pi(T) = S(T)$, the by no-arbitrage $\pi(t) = S(t)$ for all t, i.e.

$$F(t) + Ke^{-r(T-t)} = S(t)$$

or

$$F(t) = S(t) - Ke^{-r(T-t)}$$

(b) $F_t = -rKe^{-r(T-t)}$

$F_S = 1$

$F_{SS} = 0$

So $\quad F_t + \frac{1}{2}\sigma^2 S^2 F_{SS} + r(SF_S - F)$

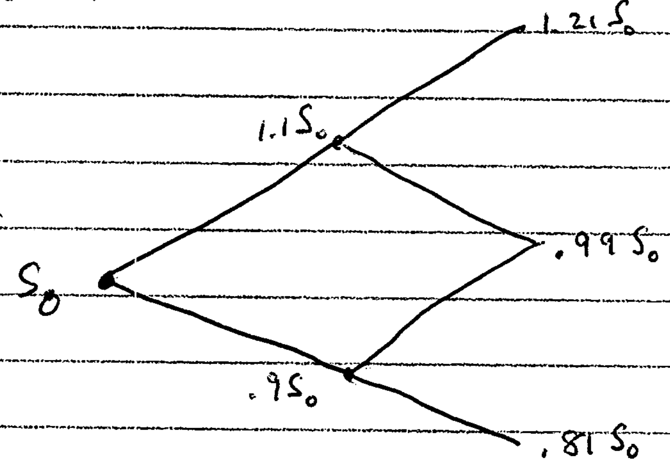
$$= -rKe^{-r(T-t)} + r\left(S - (S - Ke^{-r(T-t)})\right)$$

$$= 0$$

which shows that F satisfies the Black-Scholes PDE.

3. (a) At $t=T$, $d_c + d_p = 1$. Since this is risk free, then it discounts backwards in time like cash. i.e.

$$d_c + d_p = e^{-r(T-t)}$$



(b) For CRR

$$p^* = \frac{e^{.025} - .9}{1.1 - .9} = \frac{1.025 - .9}{.2} = \frac{.125}{.2}$$

$= .625$

$q^* = .375$

For $K = S_0$, d_c has payout only for $S_2 = u^2 S_0$ and d_p has payout for the other 2 outcomes

$$d_c = e^{-2r\,dt} E^*[d_c(T)]$$

$$= e^{-.05} p^{*2}$$

$$= .95 \times (.625)^2$$

$$= .372$$

$$
\begin{aligned}
d_p &= e^{-2r\,dt}\, E^*[\,d_p(T)\,] \\
&= e^{-2r\,dt}\left(2p^*q^* + q^{*2}\right) \\
&= e^{-2r\,dt}\left(1 - p^{*2}\right) \\
&= e^{-.05}\left(1 - (.625)^2\right) \\
&= .95\left(1 - .391\right) \\
&= .578
\end{aligned}
$$

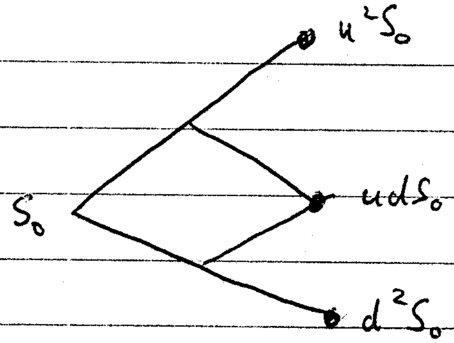
(c) Check that for $t=0$

$$d_p + d_c = e^{-rT}$$

i.e.

$$.372 + .578 = e^{-.05} = .95 \quad \checkmark$$

4. Exercise if $S_2 > K$



(a) Real probability of exercise

$$p_e = \begin{cases} 0 & \text{if } K > u^2 S_0 \\ p^2 & \text{if } udS_0 < K < d^2 S_0 \\ 1-q^2 & \text{if } d^2 S_0 < K < udS_0 \\ 1 & \text{if } K < d^2 S_0 \end{cases}$$

(b)
$$p_e^* = \begin{cases} 0 & \text{if } K > u^2 S_0 \\ p^{*2} & \text{if } udS_0 < K < d^2 S_0 \\ 1-q^{*2} & \text{if } d^2 S_0 < K < udS_0 \\ 1 & \text{if } K < d^2 S_0 \end{cases}$$

(c) If there's a risk premium, then $p > p^*$ and $q < q^*$

So $\quad p^2 > p^{*2}$

$\quad 1-q^2 > 1-q^{*2}$

$\Rightarrow \quad p_e \geq p_e^*$ in all 4 cases

(d) For three step model

$$p_e = \begin{cases} 0 \\ p^3 \\ p^3 + 3p^2 q = p^3 + 3p^2(1-p) = 3p^2 - 2p^3 \\ 1-q^3 \\ 1 \end{cases}$$

As before $p^3 > p_*^3$, $\quad 1-q^3 > 1-q_*^3$

Finally, the function $f(p) = 3p^2 - 2p^3$ has derivative $f'(p) = 6p - 6p^2 \geq 0$ ~~with~~ for $0 \leq p \leq 1$. So $f(p) \geq f(p^*)$ since $p > p^*$. So

$$3p^2 - 2p^3 \geq 3p_*^2 - 2p_*^3.$$