Math 181 Final Solutions
Winter 2007

1. (a) $p = X e^{-rT} N(-d_2) - S_0 N(-d_1)$ $\qquad T=1,\ X=100,\ S_0=110$

$\sigma = .1,\ r = .05$

$$d_1 = \frac{\log(S_0/X) + (r+\sigma^2/2)T}{\sigma\sqrt{T}}$$

$$= \frac{\log(1.1) + (.05 + .01/2)}{.1}$$

$$= \frac{(.095 + .055)}{.1} = 1.503$$

$$d_2 = d_1 - \sigma\sqrt{T} = 1.503 - .1 = 1.403$$

$N(-d_1) = .067$ $\qquad e^{-rT} = e^{-.05} = .951$

$N(-d_2) = .081$

$$p = 100 \cdot .951 \cdot .081 - 110 \cdot .067$$
$$= 7.705 - 7.37$$
$$= .33$$

(b) $\Delta = N(d_1) - 1 = N(1.5) - 1 = .933 - 1$

$= -.067$

2. $f_2 = S^2 e^{(r+\sigma^2)(T-t)}$

(a) $-\partial_t f_2 = (r+\sigma^2) f$

$\partial_S f_2 = 2S^{-1}f$

$\partial_S^2 f_2 = 2S^{-2}f$

So

$$\tfrac{1}{2}\sigma^2 S^2 f_{SS} + rSf_S - rf = (\tfrac{1}{2}\sigma^2 S^2)(2S^{-2})f + r(2S^{-1})Sf - rf$$
$$= (\sigma^2 + 2r - r)f$$
$$= (\sigma^2 + r)f$$
$$= -\partial_t f_2$$

which is the B-S equation

(b) ~~At t=T~~ payout $= f(t=T) = S^2$

(c) Set $\Delta_2 = \frac{\partial}{\partial S} f_2 = 2S^{-1}f$

π ~~be~~ is Δ-neutral

$\iff \Delta_\pi = 0$

$\iff \Delta_{f_2} - \alpha\,\Delta_S = 0 \qquad \Delta_S = 1$

$\iff 2S^{-1}f - \alpha = 0$

$$\alpha = 2S^{-1}f = 2S\,e^{(r+\sigma^2)(T-t)}$$

(d) At $t=T$

$$c_2 - p_2 = \max(S^2 - X, 0) - \max(0, X - S^2)$$
$$= S^2 - X$$

Option with payout ~~in~~ S^2 is $f(S,t)$ at t

Cash X at T is $Xe^{-r(T-t)}$ at t

So by no-arbitrage

$$c_2 - p_2 = f_2 - Xe^{-r(T-t)}$$

3. (a) Since $e^{r\,dt} > 1 > d$ and $e^{\mu\,dt} > 1 > d$, then

$p > 0$, $p' > 0$.

The constraint $1 > p$, $1 > p'$ is then

$$u > e^{r\,dt}$$
$$u > e^{\mu\,dt}$$

Since $\mu > r$, we only need to the second constraint, i.e.

$$u > e^{\mu\,dt}$$

Since $u = e^{\sigma\sqrt{dt}}$, this becomes

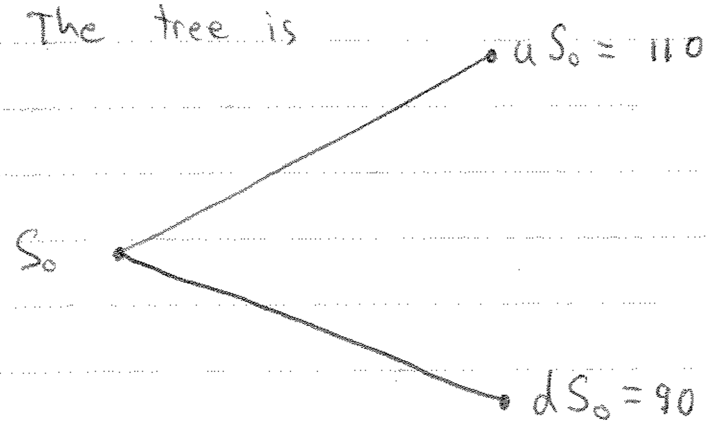
$$\sigma\sqrt{dt} > \mu\,dt$$

or

$$\sigma > \mu\sqrt{dt}$$

(b) The parameters σ and μ come from the market and cannot be changed. The parameter dt is a modeling choice. So if $\sigma < \mu\sqrt{dt}$, then just make dt smaller.

4. The tree is



The risk neutral probability is $\qquad r = .05$

$$p = (e^{r\,dt} - d)/(u-d) \qquad dt = 1,\quad u = 1.1,\quad d = .9$$
$$= (e^{.05} - .9)/(1.1 - .9)$$
$$= .151/.2$$
$$= .756$$

The put has value $(X = 102)$ at final time

$$p = \begin{cases} 0 & \text{at } S_1 = uS_0 = 110 \\ 102 - 90 = 12 & \text{at } S_1 = dS_0 = 90 \end{cases}$$

At initial node

Value of early exercise is $102 - 100 = 2$

expected value deferred exercise is

$$e^{-r\,dt}\bar{E}[p(T)] = e^{-.05}(p \cdot 0 + (1-p) \cdot 12)$$
$$= e^{-.05}(1 - .756) \cdot 12$$
$$= 2.785$$

So no early exercise

$P = 2.785$