Math 181 Midterm Solutions

Fall 2003

1. The BS eqtn is

$$f_t + \frac{1}{2}\sigma^2 S^2 f_{SS} + rSf_S - rf$$

Look for

$$f(S,t) = S^2 e^{\alpha(T-t)}$$

This satisfies the payout condition

$$S^2 = f(S,T)$$

For $t<T$,

$$f_t = -\alpha S^2 e^{\alpha(T-t)} \qquad f_t = -\alpha f$$

$$f_S = 2S e^{\alpha(T-t)} \qquad Sf_S = 2f$$

$$f_{SS} = 2e^{\alpha(T-t)} \qquad S^2 f_{SS} = 2f$$

Insert in BS and cancel f to get

$$0 = -\alpha + \sigma^2 + 2r - r$$

$$= -\alpha + \sigma^2 + r$$

$$\alpha = \sigma^2 + r$$

2. If $f(S,T) > g(S,T)$ for all S, then by no-arbitrage

$$f(S,t) > g(S,t) \quad \text{for all } S, t$$

3 (a) For the binomial model, after 4 steps S has the values

$$146,\ 120,\ 98,\ 80,\ 65.6\ .$$

The put payout is 0 for all but the last of these corresponding to 4 down steps. For this it is $75-66=9.4$

The risk neutral probability (for an up step) is

$$p = \frac{e^{.02/4}-.9}{1.1-.9} = \frac{1.005-.9}{.2} \cong .525$$

S_0

$$P_0 = e^{-r_0}\tilde{E}[\hat{P}_{payout}]$$

$$= e^{-.02}(1-.525)^4\ 9.4$$

$$=$$

$$= .469$$

(b) $$p_0 = Xe^{-r}N(-d_2) - S_0N(-d_1)$$

$$= 75e^{-.02}N(-d_2) - 100N(-d_1)$$

$$d_1 = \left(\log\left(\tfrac{100}{75}\right) + (.02 + .16/2)\right)/.4$$

$$= (.287 + .1)/.4$$

$$= .969$$

$$d_2 = d_1 - .4 = .569$$

$$N(-d_1) = .166 \qquad N(-d_2) = .2843$$

$$p_0 = 75 \cdot .98 \cdot .284 - 100 \cdot .166 = 20.8 - 16.6$$

$$= 4.2$$

4. $X_4 = 0$ requires an equal number of up and down steps. The possibilities are .

$$(++--),\ (+-+-),\ (+--+)$$
$$(--++),\ (-+-+),\ (-++-)$$

This is 6 out of 16 equally likely paths, so the probability is

$$p = \frac{6}{16} = \frac{3}{8}.$$