Mathematics of Finance

1 Lecture 3. Discrete Random Walks

3.1 Definition and Distribution

A discrete random walk is the path produced by a sequence of unit steps at discrete times. The unit steps can be positive or negative, and the discrete times are taken to be integers (in some time unit, such as days).

Denote

$$\begin{aligned} t_n &= n\Delta t = \text{discrete times} && (1) \\ x_n &= \text{position at time } t_n && (2) \\ d_n &= \text{step at time } t_n && (3) \\ &= x_n - x_{n-1}. && (4) \end{aligned}$$

We assume that the initial position is $x_0 = 0$ and that d_n is simple binomial, i.e.

$$d_n = \begin{cases} 1 & \text{prob} = \frac{1}{2} \\ -1 & \text{prob} = \frac{1}{2} \end{cases}$$

Now

$$\begin{aligned} x_n &= \sum_{i=1}^{n} d_i && (5) \\ &= \#(d_i = 1) - \#(d_i = -1) && (6) \\ &= \#(d_i = 1) - (n - \#(d_i = 1)) && (7) \\ &= 2 \cdot \#(d_i = 1) - n && (8) \\ &= 2k - n && (9) \end{aligned}$$

with

$$k = \#(d_i = 1, i \le n). \qquad (10)$$

Now every sequence of length n consisting of 1 and -1's is equally likely, with probability 2^{-n}. The probability of a value of k is just the (number of such sequences with k terms that are 1) $\cdot 2^{-n}$. This is a standard combinatorial problem

$$p(\#(d_i = 1) = k) = \left(\begin{array}{c} n \\ k \end{array} \right)$$

in which

$$\begin{aligned} \left(\begin{array}{c} n \\ k \end{array} \right) &= \#(\text{ways of choosing } k \text{ items from a sequence of length } n) &(11)\\ &= \frac{n!}{k!(n-k)!} &(12)\\ &= \frac{n \cdot (n-1) \cdot \ \cdots \ \cdot (n-k+1)}{1 \cdot 2 \cdot \ \cdots \ \cdot k} &(13)\\ & &(14) \end{aligned}$$

In summary,

$$p(x_n = 2k - n) = 2^{-n} \left(\begin{array}{c} n \\ k \end{array} \right) \text{ for } 0 \le k \le n$$

Rewrite by setting $m = 2k - n$, then $k = \frac{m+n}{2}$

$$p(x_n = m) = 2^{-n} \left(\begin{array}{c} n \\ \frac{m+n}{2} \end{array} \right)$$

for $-n \le m \le n$ with $m + n$ even.

If the probabilities are p for step $d = 1$ and $q = 1 - p$ for step $d = -1$ then this fomula is changed to

$$p(x_n = m) = \left(\begin{array}{c} n \\ \frac{m+n}{2} \end{array} \right) p^{(n+m)/2} q^{(n-m)/2}.$$

3.2 Properties of random walks

Random walks take place on a tree in x or t, as shown in Figure 1.

Possible values of x_n range from $-n$ to n in intervals of 2, i.e.

$$x_n \in \{-n, -n+2, -n+4, \ldots, n-4, n-2, n\}$$

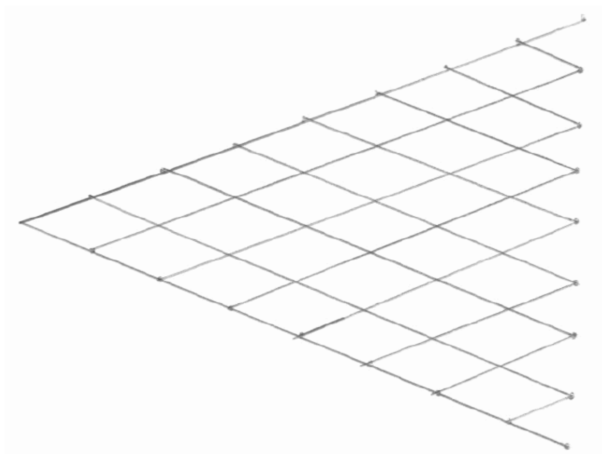

Figure 1: Tree structure of a binary random walk.

Thus the range of possible values for x_n grows like n.

The range of *likely* values grows at a smaller rate $\sqrt{n}$ for n large, as shown next.

For the steps d_i

$$\begin{aligned}
\bar{d} = E(d) &= \frac{1}{2}(1) + \frac{1}{2}(-1) = 0 \\
\mathrm{Var}(d) &= \overline{d^2} - \bar{d}^2 = \overline{d^2} = \frac{1}{2}(1)^2 + \frac{1}{2}(-1)^2 = 1
\end{aligned}$$

Apply the CLT to the sum

$$x_n = \sum_{i=1}^{n} d_i$$

of IID rv's. The normalized variable is

$$\tilde{x}_n = \frac{1}{\sqrt{n}} x_n.$$

for z a standard normal rv. Then

$$x_n \approx \sqrt{n} z.$$

This says that x_n is of size $\sqrt{n}$, instead of size n!

The reason for this is that many of the $+1$'s and -1's in the sum of x_n partly cancel, leaving something of size $\sqrt{n}$ almost all of the time. This gives the following picture