

## American Options (Hull 7.4, 7.5)

American options can be exercised at any time up to and including the expiration date. For an American call or put, the decision to exercise or hold at any time  $t$  depends just on the time value  $t$  and the underlying stock value  $S(t)$ . The exercise time  $\tau$  is chosen to maximize the value of the option.

For an American call (on a stock without dividends), early exercise is never optimal. The reason is that exercise requires payment of the strike price  $X$ . By holding onto  $X$  until the expiration time, the option holder saves the interest on  $X$ .

To see this mathematically, consider two portfolios

$$\begin{aligned} E : & \quad \text{one American call } c, Xe^{-r(T-t)} \text{ cash} \\ F : & \quad \text{one share } S. \end{aligned}$$

If the exercise time is  $\tau < T$ , the value of  $E$  is

$$\begin{aligned} E &= (S - X) + Xe^{-r(T-\tau)} \\ &< S = F. \end{aligned}$$

If the exercise is at  $\tau = T$ , then

$$\begin{aligned} E &= \max(S - X, 0) + X \\ &= \max(S, X) \\ &\geq S = F. \end{aligned}$$

It follows that  $E \geq F$  for all times, so that one should never take  $\tau < T$ .

For an American put (or an American call on a stock with dividends) early exercise is sometimes optimal. Suppose for example, that the stock price  $S$  falls to nearly 0. Then the option holder stands to gain more by exercise than by waiting. The reason is that the payout  $X - S$  cannot increase much, but by early exercise, the option holder will get the interest on the payout.

Since the early exercise decision for an America depends only on  $t$  and  $S(t)$ , there is an early exercise boundary  $S = B(t)$ . For  $S(t) > B(t)$ , it is better to hold on to the option. If the stock price hits  $S(t) = B(t)$ , then the

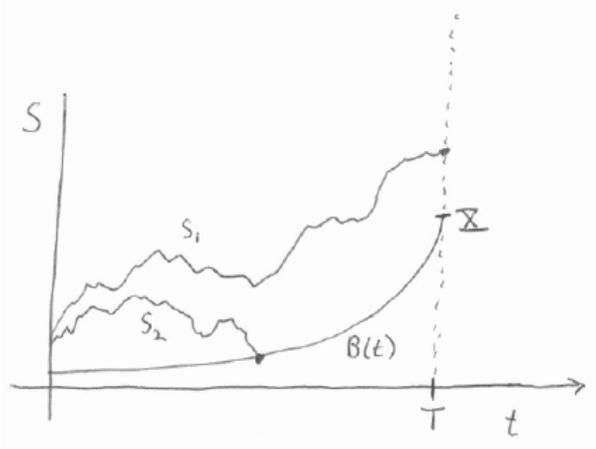


Figure 1: Early exercise boundary  $S = B(T)$  and two paths  $S_1$  (with early exercise) and  $S_2$  (without early exercise) for an American option.

put option should be exercised. (If  $S(0) < B(0)$ , then the option should be exercised when first issued).

Figure 1 shows the early exercise boundary for an American put  $S = B(t)$ . Two typical paths are also drawn. The first ( $S_1$ ) will be exercised at the time that it hits the early exercise boundary; the second ( $S_2$ ) never hits the early exercise boundary so that it is not exercised.

Another interesting feature, is that the slope  $B'(T)$  of the boundary is infinite at the expiration time. It's local behavior is given by

$$B(t) = c\sqrt{(T-t)\log(T-t)}.$$