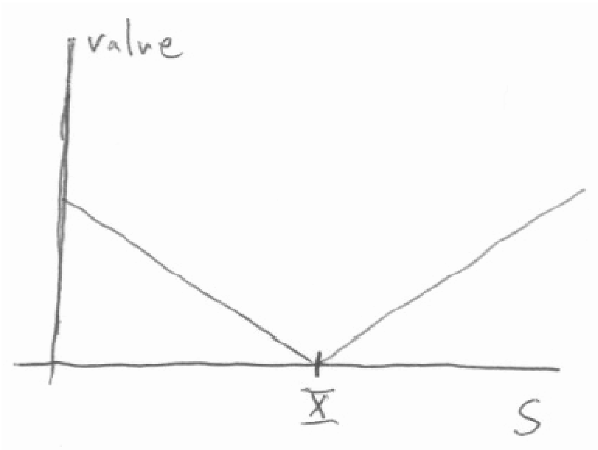

Figure 1: Payoff for a straddle option, as a function of $S = S(T)$.

Math 181 Lecture 17

Exotic Options

Derivatives that are more complicated than standard European and American calls and puts are called exotic options. Here we describe only a few of the common exotic options.

Combination Options. These are options consisting of a combination of puts and calls

1. A *straddle* is a combination of a put and a call. It's payoff at expiration is

$$\begin{aligned} f_{\text{straddle}}(S_T, T) &= p(S_T, T) + c(S_T, T) \\ &= \max(0, X - S_T) + \max(0, S_T - X) \\ &= \max(X - S_T, S_T - X) \\ &= |X - S_T| \end{aligned}$$

which is plotted if Figure 1.

It's also useful to plot the profit from a straddle, which is its payoff minus its cost (suitably discounted). Projected at time T

$$\text{cost} = e^{rT} f_{\text{straddle}}(S_0, 0).$$

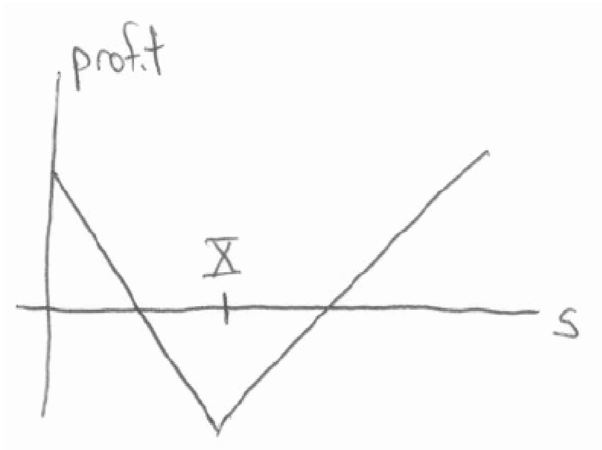

Figure 2: Profit for a straddle option, as a function of $S = S(T)$.

This is a constant with respect to the random variations of S_T. So the profit is

$$\text{profit} = f(S_T, T) - \text{cost}$$

as plotted if Figure 2.

A straddle is used by an investor who expects a large move in the stock. Since a straddle equals a put plus a call, the Black-Scholes gives its price as

$$f_{\text{straddle}}(S_0, 0) = S(N(d_1) - N(-d_1)) - Xe^{-rT}(N(d_2) - N(-d_2)).$$

2. Butterfly.

A butterfly option has the following payoff and profit (Figure 3).

A butterfly can be made by buying one call option and selling two others

$$f_{bfly} = \begin{cases} 0 & S_T < X_1 \\ S_T - X_1 & X_1 < S_T < X_2 \\ X_3 - S_T & X_2 < S_T < X_3 \\ 0 & X_3 < S_T \end{cases} \tag{1}$$

in which X_2 is usually chosen to make the payout function continuous in S.

Binary Options.

These are options with discontinuous payoffs.

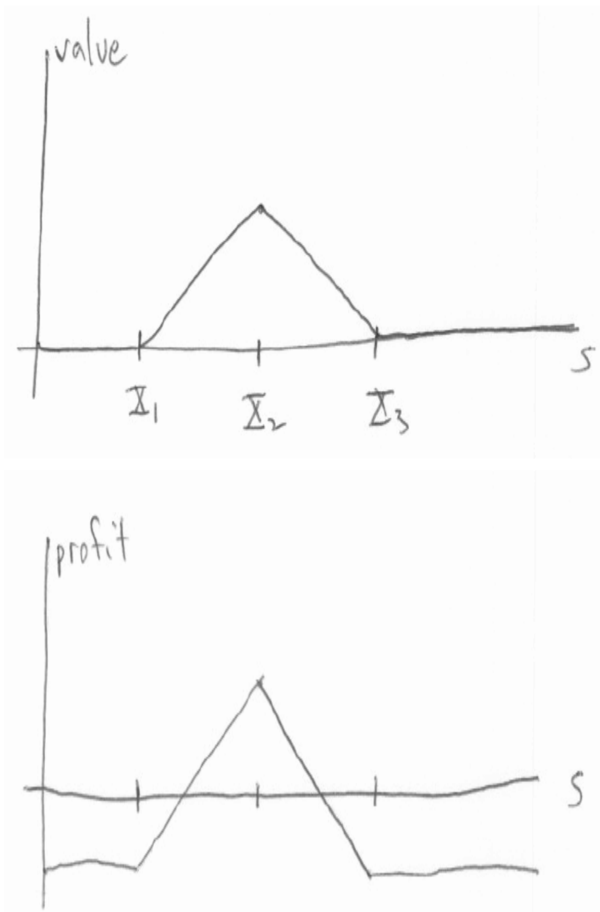

Figure 3: Payout (top) and profit (bottom) for a butterfly option, as a function of $S = S(T)$.

3. Asset-or-Nothing.

The payoff is

$$f_{aon} = \begin{cases} 0 & S_T < X \\ S_T & S_T > X \end{cases}.$$

It's value is

$$f_{aor}(S_0, 0) = S_0 e^{-rT} N(d_1).$$

Path-Dependent Options.

The payoff at expiration for a path-dependent option depends on the whole evolution $S(t)$ for $0 < t < T$, rather than just the final value $S(T)$. They are much more difficult to price.

4. Lookback Option.

The payoff of a lookback at expiration T is equal to the maximum of the stock price over the period $t \in [T - \bar{T}, T]$. If $\bar{T} = T$ so that the payoff is the average over the entire period of the option, then there is an explicit formula for the price of a lookback.

5. Asian Option.

The payoff of an Asian option depends on the average price over some period

$$S_{av} = \bar{T}^{-1} \int_{T-\bar{T}}^{T} S(t)dt.$$

The payoff for an Asian call and put is

$$\begin{aligned} f_{ac} &= \max(0, S_{av} - X) \\ f_{ap} &= \max(0, X - S_{av}). \end{aligned}$$

Monte Carlo Pricing of Exotic Options.

Many options can be priced using combinations of the Black-Scholes formulas for vanilla puts and calls. Others can be priced by solving the Black-Scholes PDEs. In other cases, however, neither of these approaches works and the only pricing method is Monte Carlo. Here we discuss Monte Carlo pricing for the options mentioned above.

6. Options depending on a single security at payoff.

For an option that depends only on the value of a single security S at the exercise time T, the Monte Carlo formula is simple. Let the value of the option be $P(S,t)$ and the payoff at time T be $f(S)$. Then by risk neutral valuation

$$\begin{aligned} P(S,t) &= e^{-r(T-t)}E[f(S(T))] \\ S(T) &= S\exp\{(r-\sigma^2/2)(T-t)+\sigma\sqrt{T-t}\;\omega\} \end{aligned}$$

in which ω is an $N(0,1)$ random variable and E means expectation with respect to ω. Note that this formula is for the risk-neutral process $S(T)$, not the real process. Once P is written as an expectation, Monte Carlo is directly applicable.

6. Options depending on the history of a single security.

Here we generalize the risk-neutral valuation formula above to the entire path of values of an underlying security.

We start with the random walk model with normal increments for the underlying stock price. The real stock price in this model is

$$S_n^{real} = S_0\exp\{(\mu-\sigma^2/2)t_n+\sigma\sqrt{dt}\sum_{k=1}^{n}\omega_k\} \tag{2}$$

in which the $\omega_k' s$ are independent $N(0,1)$ random variables. The corresponding risk neutral process is obtained by just replacing the real average growth rate r by the risk-free rate of return r to obtain

$$S_n = S_0\exp\{(r-\sigma^2/2)t_n+\sigma\sqrt{dt}\sum_{k=1}^{n}\omega_k\}. \tag{3}$$

Now suppose that the payoff function is

$$f = f(S_1,S_2,\cdots,S_N) \tag{4}$$

in which $T=t_N$ is the expiration time. Then the present value at time 0 is

$$P(S,0) = e^{-rT}E[f = f(S_1,S_2,\cdots,S_N)] \tag{5}$$

in which the expectation is over the N random variables $\omega_k,\ 1\le k\le N$. Be sure to use the risk-neutral formulas with μ replaced by r for S_n.