Math 181   Lecture 15

Hedging and the Greeks   (Chap. 14, Hull)

One use of derivation is for investors or investment banks to manage the risk of their investments. If an investor buys a stock for price S_0, they may also buy a put at strike X. Then the most they could lose on the stock would be $S_0 - X$.

A bank that sells an option is also interested in hedging the option to limit their risk. Suppose that a bank sells a call. If they do nothing else, this is *naked position*. If they buy shares of the stock, so that they can satisfy the option if it is exercised, this is a *covered position*. Neither of these is very satisfactory. If the stock price goes up, the bank in a naked position will have to buy it at its high price to satisfy the exercised option. If the stock price goes down, the option will not be exercised, and the bank in the covered position will absorb this loss.

A tempting, but flawed method to eliminate risk, is the *stop-loss strategy*. The idea is to buy the stock whenever its price rises above X and sell it whenever it falls below X. The cost of this hedging strategy is (incorrectly) calculated to be

$$Q = \max[S(0) - X, 0]$$

by the following method:

If $S(0) > X$, then the stock must be bought initially. When the price falls below X the stock is sold at price X, the cost being $S(0) - X$ so far. From then on the strategy has no cost since the stock is always bought and sold at price X (including the exercise of the option). If the price is $S(0) < X$, then the cost is 0, for this same reason. This gives Q.

This reasoning is incorrect for several reasons. First, the buying and selling of the stock occurs at different times and must be discounted. Second, since transactions are not instantaneous, they cannot be made exactly at X. Purchases may be made at $X + \delta$ and sales at $X - \delta$ with a net loss. As one tries to make δ smaller, the number of times that the stock value crosses $X - \delta$ to $X + \delta$ (or vice versa) gets larger and larger. The effect is a net loss.

The preferred hedging schemes use derivatives of the option price, which are referred to as "the Greeks". Here are definitions of the usual Greeks for

an option f

$$
\begin{aligned}
\text{delta:} \quad & \Delta = \frac{\partial f}{\partial s} \\
\text{gamma:} \quad & \Gamma = \frac{\partial^2 f}{\partial s^2} \\
\text{theta:} \quad & \Theta = \frac{\partial f}{\partial t} \\
\text{vega:} \quad & v = \frac{\partial f}{\partial \sigma} \\
\text{rho:} \quad & \rho = \frac{\partial f}{\partial r}.
\end{aligned}
\tag{1}
$$

For a call option

$$
c = SN(d_1) - Xe^{-r(T-t)}N(d_2)
$$

and the Greeks are

$$
\begin{aligned}
\Delta = \frac{\partial c}{\partial s} &= N(d_1) + SN'(d_1)\frac{\partial d_1}{\partial s} - Xe^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial s} \\
&= N(d_1) + \{SN'(d_1) - Xe^{-r(T-t)}N'(d_2)\}\frac{\partial d_1}{\partial s}
\end{aligned}
$$

since $d_2 = d_1 - \sigma\sqrt{T-t}$ so that $\frac{\partial d_2}{\partial s} = \frac{\partial d_1}{\partial s}$. Now $N'(d) = \frac{e^{-d^2/2}}{\sqrt{2\pi}}$ and

$$
\begin{aligned}
N'(d_1) &= \exp{-d_1^2/2}/\sqrt{2\pi} \\
N'(d_2) &= \exp{-(d_1 - \sigma\sqrt{T-t})^2/2}/\sqrt{2\pi} \\
&= \frac{1}{\sqrt{2\pi}}e^{-d_1^2/2}\exp{\sigma\sqrt{T-t}d_1 - \sigma^2(T-t)/2} \\
&= N'(d_1)\exp\{\log(S/x) + (r + \sigma^2/2)(T-t) - \sigma^2(T-t)/2\} \\
&= N'(d_1)(S/X)e^{r(T-t)}.
\end{aligned}
$$

So

$$
SN'(d_1) = Xe^{-r(T-t)}N'(d_2)
$$

and for a call

$$
\Delta = N(d_1).
$$

Next

$$
\Gamma = \frac{\partial^2 c}{\partial s^2} = \frac{2}{\partial s}\left(\frac{\partial c}{\partial s}\right) = \frac{\partial}{\partial s}\Delta
$$

$$\begin{aligned}
&= N'(d_1)\frac{\partial d_1}{\partial s} \\
&= N'(d_1)\frac{1}{s}\frac{1}{\sigma\sqrt{T-t}} \\
&= \frac{N'(d_1)}{s\sigma\sqrt{T-t}}.
\end{aligned}$$

Note that Δ and Γ are positive for a call.

Similar calculations show that

$$\begin{aligned}
\Theta &= -\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} - rXe^{-r(T-t)}N(d_2) \\
v &= s\sqrt{T-t}N'(d_1) \\
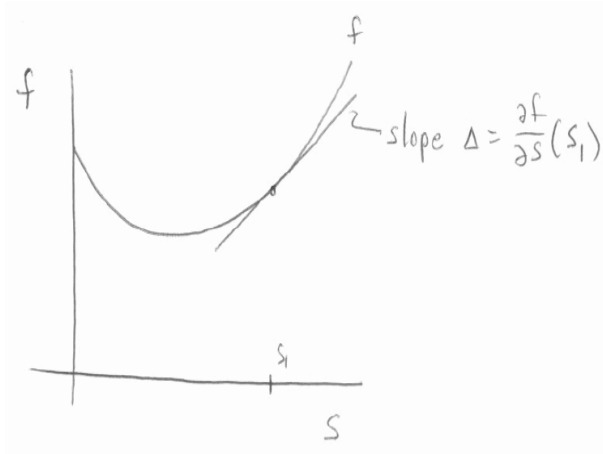\rho &= X(T-t)e^{-r(T-t)}N(d_2).
\end{aligned}$$

Figure 1: Graph of a hypothetical option f and the tangent line of slope $\Delta = \partial f/\partial S(S_1)$ at a point $S = S_1$.

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Hedging and the Greeks (continued)

The first type of hedging using the Greeks is *delta hedging*. This is just what was done to make the Black-Scholes argument. In a delta hedge, for each option one also buys $-\Delta = -\frac{\partial f}{\partial S}$ shares of the stock. The hedge portfolio is

$$A \quad : \quad \alpha \text{ options}, \ -\alpha\Delta = -\alpha\frac{\partial f}{\partial S} \text{ shares of stock}$$

with value

$$A \ = \ \alpha f - \alpha\Delta S.$$

Graphically, Δ is the slope of the tangent line to f, as shown in Figure 1.

Then $f - \delta S$ is shown in Figure 2.

This shows that for any option whose value f is convex, then the delta hedged portfolio is everywhere above the current price.

Note that the Δ of A is

$$\Delta_A = \frac{\partial A}{\partial S} \ = \ \alpha\frac{\partial f}{\partial S} - \alpha\Delta\frac{\partial}{\partial S}(S)$$
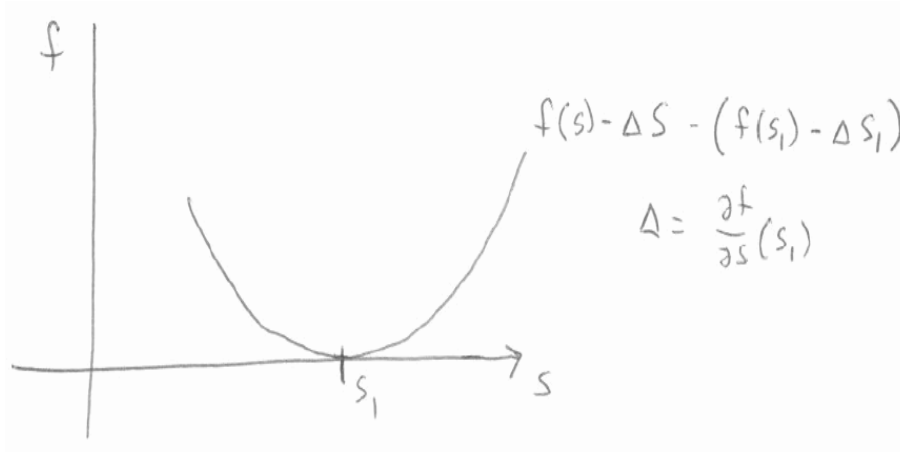$$= \ \alpha\frac{\partial f}{\partial S} - \alpha\Delta = 0.$$

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Figure 2: Graph of a $f - \Delta S$ (shifted by a constant amount $f(S_1) - \Delta S_1$) vs. S.

A portfolio where Δ is zero is called *delta neutral.*

A delta-hedged portfolio has no risk for infinitesimal movements, but it does require that the value of Δ is changed as S changes. Moreover there is risk for large changes in S.

For a delta-hedged portfolio, there are still changes in value for the following reasons: the value changes in time and Δ changes in time. The first of these is measured by Θ and the second by Γ.

In fact we can expand the value of an option f or portfolio π by

$$
\begin{aligned}
d\pi &= \pi(t+dt,\ S+dS) - \pi(t,S) \\
&= \pi_s dS + \frac{1}{2}\pi_{ss} dS^2 + \pi_t dt + \cdots \\
&\approx \Delta dS + \frac{1}{2}\Gamma dS^2 + \Theta dt.
\end{aligned}
$$

If the portfolio is delta-neutral, then $\Delta = 0$ so that

$$
d\pi \approx \frac{1}{2}\Gamma dS^2 + \Theta dt.
$$

The change in time is expected and cannot be hedged away. The portfolio can also be made Γ neutral. This is a little subtle, however, because the stock itself, with value S, has $\Gamma = 0$. So we needed to add a traded option

where Γ value is Γ_T to the portfolio. Choose to add w_T of the traded option, in which

$$w_T \Gamma_T + \Gamma = 0.$$

One of the assumptions of Black-Scholes is that the volatility σ is constant. Actually, volatilities change in time. Vega $v = \frac{\partial}{\partial \sigma} f$ measures the sensitivity of option price to changes in σ. Again this can be hedged away by using a traded option with known vega.

Similarly $\rho = \frac{\partial}{\partial r} f$ measures sensitivity of option price to changes in r. Again, this can be hedged away using a traded option.