

## Risk-Neutral Valuation

The most surprising feature of the Black-Scholes equation, or of the Black-Scholes formulas, is that they do not involve the average rate of return  $\mu$ . No matter what value  $\mu$  takes, the value of the option is the same for given values of  $\sigma$  and  $r$ . This has led to a method called “risk-neutral valuation” for finding option prices. We explain this first in a descriptive way and then in a mathematical way.

In the real world  $\mu > r$  if  $\sigma > 0$ ; since investors are risk-averse, they require a greater rate of return to assume the volatility  $\sigma$ . As discussed in the binomial tree sections, this is the reason that the option price is not the expected value

$$f(S, t) \neq e^{-r(T-t)} E(f(S_T, T)). \quad (1)$$

We may imagine a world in which all investors are risk-neutral. That is, in which  $\mu = r$  for any  $\sigma$ . In the imaginary “risk-neutral world”, the option price is the average, i.e.

$$f(S, t) = e^{-r(T-t)} \bar{E}(f(S_T, T)) \quad (2)$$

in which  $\bar{E}$  is the average in the risk-neutral world. Moreover, Black-Scholes showed that the option price is the same in the real world and in the risk-neutral world. So (2) gives the correct price in the real world, using an average in the risk-neutral world. This pricing method is referred to as “risk-neutral valuation”.

In order to make the average in (2) explicit, we need a probabilistic representation of  $S(t)$ . The Gaussian random walk for  $S$  is

$$dS_{n+1} = \mu S_n dt + \sigma S_n \sqrt{dt} \nu_{n+1}.$$

Now consider  $\log S_{n+1} = y_{n+1} = Y(S_{n+1})$  with

$$Y(S) = \log S, \quad Y_s = S^{-1}, \quad Y_{ss} = -S^{-2}.$$

Now expand

$$\begin{aligned}
dy_{n+1} &= Y(S_{n+1}) - Y(S_n) \\
&\cong Y_s dS_{n+1} + \frac{1}{2} Y_{ss} (dS_{n+1})^2 \\
&= S_n^{-1} dS_{n+1} - \frac{1}{2} S_n^{-2} (dS_{n+1})^2 \\
&= S_n^{-1} (\mu S_n dt + \sigma S_n \sqrt{dt} \nu_{n+1}) \\
&\quad - \frac{1}{2} S_n^{-2} (\mu S_n dt + \sigma S_n \sqrt{dt} \nu_{n+1})^2 \\
&= \mu dt + \sigma \sqrt{dt} \nu_{n+1} - \frac{1}{2} (\mu dt + \sigma \sqrt{dt} \nu_{n+1})^2 \\
&= \mu dt + \sigma \sqrt{dt} \nu_{n+1} - \frac{1}{2} \sigma^2 \nu_{n+1}^2 dt + \mathcal{O}(dt^{3/2}) \\
&\cong \mu dt + \sigma \sqrt{dt} \nu_{n+1} - \frac{1}{2} \sigma^2 dt
\end{aligned}$$

as in the Black-Scholes derivation. Therefore

$$dy_{n+1} = \tilde{\mu} dt + \sigma \sqrt{dt} \nu_{n+1}$$

in which

$$\tilde{\mu} = \mu - \frac{1}{2} \sigma^2. \quad (3)$$

Add this up to get

$$\begin{aligned}
y_{n+1} &= (y_{n+1} - y_n) + (y_n - y_{n-1}) + \cdots + (y_1 - y_0) + y_0 \\
&= y_0 + \sum_{k=1}^{n+1} dy_k \\
&= y_0 + \tilde{\mu}(n+1)dt + \sigma \sqrt{dt} \sum_{k=1}^{n+1} \nu_{n+1} \\
&= y_0 + \tilde{\mu} t_{n+1} + \sigma \sqrt{t_{n+1}} \omega
\end{aligned}$$

in which  $\omega$  is an  $N(0, 1)$  random variable. This used the additive property of Gaussian random variables.

This says that

$$\log S(t_{n+1}) = \log S(0) + \tilde{\mu} t_{n+1} + \sigma \sqrt{t_{n+1}} \omega.$$

Apply exp to obtain

$$S(t_{n+1}) = S(0) e^{\tilde{\mu} t_{n+1} + \sigma \sqrt{t_{n+1}} \omega}.$$

Replace  $t_{n+1}$  by  $t$  and  $\tilde{\mu}$  by  $\mu - \sigma^2/2$

$$S(t) = S(0)e^{(\mu - \sigma^2/2)t + \sigma\sqrt{t} \omega}. \quad (4)$$

This formula for  $S(t)$  gives a probabilistic representation, since  $\omega$  is an  $N(0, 1)$  random variable. A slight extension is

$$S(T) = S(t)e^{(\mu - \sigma^2/2)(T-t) + \sigma\sqrt{T-t} \omega} \quad (5)$$

for any  $T > t$ .

The formulas (4), (5) are for the real process, since they involve the growth rate  $\mu$ , including the risk premium. In the risk-neutral world, the only difference is that  $\mu$  is replaced by  $r$ . This says that the risk neutral representation for  $S(t)$  given  $S(\bar{t})$  is

$$S(T) = S(t)e^{(r - \sigma^2/2)(T-t) + \sigma\sqrt{T-t} \omega}. \quad (6)$$

In the risk neutral valuation formula (2), this risk-neutral stock price is used, i.e.

$$f(S, t) = e^{-r(T-t)} E_{\omega}[f]$$

in which

$$f = f(S \exp\{(r - \sigma^2/2)(T - t) + \sigma\sqrt{T - t} \omega\}).$$