Math 181 Lecture 13

Price of Calls and Puts from Black-Scholes

The Black-Scholes equation for the price $f = f(t, S)$ of an option is

$$-f_t = \frac{1}{2}\sigma^2 S^2 f_{ss} + fSf_s - rf. \tag{1}$$

In our stock price model, the value of S can never become 0 or negative. So this equation is to be solved on $S > 0$.

This is a general equation that is valid for any option. How do we insert information about the specific option we wish to price? Through the "initial conditions", as shown next.

Consider a call with price $f = c(S, t)$. The only information we have about c is that

$$c(S, T) = \max(0, S - X)$$

at the expiration time $t = T$. We wish to find the price for $t < T$. The resulting equation is

$$-c_t = \frac{1}{2}\sigma^2 S^2 c_{ss} + rSc_s - rc \quad \text{for } t < T \tag{2}$$

$$c = \max(0, S - X) \qquad \text{for } t = T. \tag{3}$$

As stated above, this is to be solved for $S > 0$.

We call the condition (3) an initial condition, in analogy to the use of initial data for differential equations. A better name might be a "final condition". It is fortunate that the data is specified at the end of the time period $t \leq T$, because this is consistent with the $-f_t$ term in (2). That is, the natural direction in (2) is *backwards* in time. So it makes sense to give data at the end and solve backwards.

Black and Scholes succeeded in solving (2) and (3). The solution is

$$c(S, t) = SN(d_1) - Xe^{-r(T-t)}N(d_2) \tag{4}$$

in which

$$\begin{aligned} d_1 &= \frac{\log(S/X) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \qquad (5) \\ d_2 &= d_1 - \sigma\sqrt{T-t} \\ &= \frac{\log(S/X) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \qquad (6) \end{aligned}$$

and $N(x)$ is the cumulative distribution function for an $N(0,1)$ random variable. N satisfies

$$\begin{aligned} N'(x) &= (2\pi)^{-\frac{1}{2}} e^{-x^2/2} \\ N \to 0 & \quad \text{as } x \to -\infty \end{aligned}$$

so that

$$\begin{aligned} N(x) &= (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-x'^2/2} dx' \qquad (7) \\ &= \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}(x/\sqrt{2}) \end{aligned}$$

in which $\mathrm{erf}(y)$ is the "error function" defined by

$$\mathrm{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-t^2} dt.$$

This is a standard function that appears in many mathematical handbooks.

The Black-Scholes formula (4) provides the price $c(S,t)$ for a call as a function of the current time t and the current stock price S. This formula can be written in the following way

$$c(S,t) = e^{-r(T-t)} \left\{ SN(d_1)e^{r(T-t)} - XN(d_2) \right\}.$$

The term $N(d_2)$ is the probability of exercise of the option, in the "risk-neutral world" that will be discussed in the next lecture. So $XN(d_2)$ is the strike price times the probability that it is paid.

The term $SN(d_1)e^{r(T-t)}$ is the expected value at $t = T$ of a variable that is $S(T)$ if $S(T) > X$ and 0 otherwise. The factor $e^{-r(T-t)}$ is the discount factor.

There is a similar formula for a put. The price p of a put satisfies

$$\begin{aligned} -p_t = \tfrac{1}{2}\sigma^2 S^2 p_{ss} + rSp_s - rp & \quad \text{for } t < T \\ p = \max(X - S, 0) & \quad \text{for } t = T. \end{aligned}$$

The solution is

$$p = Xe^{-r(T-t)}N(-d_2) - SN(-d_1). \qquad (8)$$