Midterm Solutions

1. (i) Since f is an option price, it satisfies the Black-Scholes equation

$$-f_t = \frac{1}{2}\sigma^2 S^2 f_{ss} + rSf_s - rf.$$

It's "initial condition" is its value at the expiration time $t = T$. At this time the holder pays an amount X and receives one share of the stock which is worth S_T, so that

$$f(S_T, T) = S_T - X. \tag{1}$$

The formula $f(S,t) = S - Xe^{-r(T-t)}$ satisfies

$$\begin{aligned} f_t &= -rXe^{-r(T-t)} = r(f - S) \\ f_S &= 1 \\ f_{SS} &= 0 \end{aligned}$$

Therefore

$$\begin{aligned} \frac{1}{2}\sigma^2 S^2 f_{ss} + rSf_s - rf &= rS - rf \\ &= -f_t \end{aligned}$$

which shows that f satisfies the Black-Scholes equations, and

$$f(S,T) = S - X$$

which shows that f satisfies the initial condition.

(ii) At expiration, $f = S - X$ whereas the value of the call is $c = \max(S - X, 0)$. In particular, $f \leq c$ at $t = T$. By no-arbitrage, it follows that $f \leq c$ for all t. In addition, since $f < c$ with some positive probability, then in fact $f < c$ for all $t < T$.

2. (i) By its definition

$$x_5 = \omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5. \tag{2}$$

Since these are each independent Gaussians with variance 1, then

$$\begin{aligned} \sigma^2(x_5) &= \sigma^2(\omega_1) + \sigma^2(\omega_2) + \sigma^2(\omega_3) + \sigma^2(\omega_4) + \sigma^2(\omega_5) \\ &= 5. \end{aligned} \tag{3}$$

(ii) Since x_5 is a Gaussian random variable with variance 5 and mean 0, then its probability density is $p(x) = (10\pi)^{-1/2} exp(-x^2/10)$. It follows that

$$\begin{aligned}
E[\max(x_5, 0)] &= \int_{-\infty}^{\infty} \max(x, 0) p(x) dx \\
&= \int_0^{\infty} x p(x) dx \\
&= \int_0^{\infty} x (10\pi)^{-1/2} exp(-x^2/10) dx \\
&= (10\pi)^{-1/2} \int_0^{\infty} \sqrt{5} y e^{-y^2/2} \sqrt{5} dy \quad \text{in which } x = \sqrt{5} y \\
&= (5/2\pi)^{1/2} \int_0^{\infty} \frac{-de^{-y^2/2}}{dy} dy \\
&= (5/2\pi)^{1/2} \left[-e^{-y^2/2} \right]_{y=0}^{y=\infty} \\
&= (5/2\pi)^{1/2}. \qquad (4)
\end{aligned}$$

3. The Black-Scholes formula for a put price with $t = 0$ is

$$\begin{aligned}
p &= X e^{-rT} N(-d_2) - S_0 N(-d_1) \\
d_1 &= \frac{\log(S/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} \\
d_2 &= d_1 - \sigma\sqrt{T}.
\end{aligned}$$

For the security in this problem

$$\begin{aligned}
S_0 &= X = 1,\ T = 4,\ r = .05,\ \sigma = .2 \\
d_1 &= \frac{0 + (.05 + .04/2)4}{.2\sqrt{4}} = .7 \\
d_2 &= .7 - .4 = .3\ .
\end{aligned}$$

From the table on the back of the exam

$$\begin{aligned}
N(-d_1) &= N(-.7) = .24 \\
N(-d_2) &= N(-.3) = .38 \\
e^{-rT} &= e^{-.2} = .82\ . \qquad (5)
\end{aligned}$$

Insert this into the Black-Scholes formula to get the call price

$$\begin{aligned} c &= .82 \times .38 - .24 \\ &= .072 \ . \end{aligned}$$

4. From put call parity, we have

$$p - c = S - Xe^{-r(T-t)}.$$

Therefore the equation $p = x$ at $S = \bar{S}(t)$ implies that

$$\bar{S}(t) = Xe^{-r(T-t)}. \tag{6}$$